Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2018

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work.

Problem 1. For which values of k and b does the following system of linear equations have (i) no solution, (ii) one unique solution, and (iii) infinitely many solutions:

$$\begin{array}{rcrcrcl} & & 2x_2 & + & 4x_3 & = & b \\ 3x_1 & + & 2x_2 & + & kx_3 & = & 7 \\ x_1 & + & x_2 & & & = & 2 \end{array}$$

Problem 2. Find a basis for the vector subspace of $\mathbb{R}^4$ orthogonal to the vector $[1 \;\; 2 \;\; 4 \;\; 3]^T$.

Problem 3. Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & -1 \\ 0 & 4 \\ 2 & -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 7 \\ -6 \\ 2 \end{bmatrix}.$$

(a) Compute the QR factorization of the matrix A.

(b) Use this factorization to determine the least squares solution of $Ax = b$.

Problem 4. Prove that the inverse of an upper triangular matrix in $\mathbb{C}^{n\times n}$ is also upper triangular. (Hint: Use induction on n.)

Problem 5. Let $A \in \mathbb{C}^{m\times n}$. Prove that the orthogonal complement of Range(A) is Null(A^T).

Problem 6. Let $A \in \mathbb{R}^{m\times n}$. Define the matrix operator

$$\Theta_p(A) = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

(a) Prove that

$$\Theta_1(A) = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{i,j}|.$$

(b) Prove that Θ_1 is a norm.

Problem 7. Let $x, y \in \mathbb{R}^n$.

(a) Find the matrix P that projects x onto y.

(b) Find the matrix $\widetilde{P}$ that projects x onto the orthogonal complement of y.

(c) Show that y is in the null space of $\widetilde{P}$.

Problem 8. Let $A \in \mathbb{C}^{n\times n}$ be skew-symmetric. Prove that the eigenvalues of A are either 0 or purely imaginary.

Problem 9. Prove that there is no matrix in $\mathbb{C}^{2\times 2}$ whose inverse is the component-wise reciprocals of the matrix, i.e., there is no 2×2 complex matrix satisfying

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{d} \end{bmatrix}.$$

Problem 10. Short answers.

(a) Let $A \in \mathbb{R}^{n\times n}$ be an orthonormal matrix. Write the singular value decomposition (SVD) for A.

(b) Suppose $\{(\lambda_i, v_i)\}_{i=1}^n$ are the eigenpairs of the matrix $A \in \mathbb{R}^{n\times n}$. If I_n is the $n \times n$ identity matrix, what are the eigenpairs corresponding to the matrix $A + 3I_n$? Explain.

(c) Suppose $u_1, u_2, u_3 \in \mathbb{R}^3$ are linearly independent with $u_1 \perp u_2$ and $u_1 \perp u_3$. True or false: $u_2 \perp u_3$. Explain.