Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2019

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work. There are 7 problems.

1. (20 pts) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   (a) Compute the singular value decomposition of A.

   (b) Find the bases for the four fundamental subspaces $C(A)$, $N(A^T)$, $C(A^T)$, and $N(A)$ associated with A. Specify the rank and nullity of A.

2. (25 pts) A is a 3×3 matrix, and we know that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}, \quad A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}.$$

   (a) What are the eigenvalues and associated eigenvectors of A? Can we use the set of eigenvectors as a basis for $\mathbb{R}^3$? Why or why not? If yes, does this basis have any special properties?

   (b) Calulate $A^{2019} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

   (c) Does the linear system $A\vec{x} = \vec{b}$ have a solution for any $\vec{b} \in \mathbb{R}^3$? If so, why? If not, for what kind of $\vec{b} \in \mathbb{R}^3$ is $A\vec{x} = \vec{b}$ solvable?

   (d) Determine whether matrix A has the following properties. Explain your reasoning.

      i. diagonalizable

      ii. invertible

      iii. orthogonal

      iv. symmetric

3. (20 pts) You are a TA for a linear algebra class. An exam question asks students to solve a specific linear system

$$A\vec{x} = \vec{b}.$$

   The answer key given by your instructor says that all solutions to the system should have the form

$$\vec{x} = \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(a) A student wrote that all solutions to the system are

$$\vec{x} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \text{with } t \text{ being any real number.}$$

Should you dock points from this student for incorrect answer? Explain why.

(b) Write down all properties of matrix A that you can deduce, such as its shape, rank, four fundamental subspaces, etc. Explain your reasoning.

(c) What can you deduce about $\vec{b}$?

4. (10 pts) Suppose both matrices $A_{m\times n}$ and $B_{n\times s}$ have full column rank. Suppose furthermore that the range of AB is the same as the range of A. What relations (e.g. $<$, $\geq$, $=$) must hold among the integers m, n, and s? Carefully explain your reasoning and then give an example that illustrates your conclusions.

5. (15 pts) Let $\mathbb{P}_k$ denote the vector space of all polynomial of degrees k or less. Define a subset S of $\mathbb{P}_2$ to be

$$S = \left\{ f \in \mathbb{P}_2 \, : \, \int_0^1 f(x)\, dx = \int_0^1 f'(x)\, dx \right\}.$$

Show that S is a linear subspace of $\mathbb{P}_2$. Determine its dimension and find a basis for S.

6. (15 pts) Let $\mathfrak{M}_{2\times 2}$ be the vector space of all 2-by-2 matrices. Let T be the transformation on $\mathfrak{M}_{2\times 2}$ that *transposes* every 2×2 matrix. That is, $T(A) = A^T$.

$$T : \mathfrak{M}_{2\times 2} \to \mathfrak{M}_{2\times 2}, \; T(A) = A^T \text{ for every } A \in \mathfrak{M}_{2\times 2}.$$

(a) Show that T is a linear transformation.

(b) Determine all the eigenvalues of T and specify their associated eigenvectors (really, eigenmatrices).

7. (15 pts) Let $A_{m\times n}$ be a real matrix.

(a) Show that A^TA is symmetric.

(b) Show that A^TA is positive semi-definite.

(c) Under what condition on A is A^TA positive definite? Explain.