Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, May 2019

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work. There are 8 problems, for a total of 100 points.

1. (15 pts: 3 each) For each of the following statements, either give a specific example, briefly justified, or explain why no such example can exist.
   
   (a) A 2-by-3 matrix $A$ in echelon form whose columns span $\mathbb{R}^2$.
   
   (b) Nonzero vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^2$ such that $\vec{w}$ is not in $\text{span}\{\vec{u}, \vec{v}\}$.
   
   (c) A linear system with two equations and three variables but does not have any solutions.
   
   (d) A 3-by-3 non-zero matrix $B$ such that the vector \[
   \begin{bmatrix}
   1 \\
   1 \\
   1 
   \end{bmatrix}
   \]
   is a solution of $B\vec{x} = \vec{0}$.
   
   (e) A matrix $C$ (of any size you want) and a vector $\vec{b}$ for which the solution set of $C\vec{x} = \vec{b}$ is a plane.

2. (10 pts: 5 each) Consider the system of equations
   \[
   \begin{align*}
   x + 2y + 3z + 4t &= b_1 \\
   x + 2y + 4z + 6t &= b_2 \\
   z + 2t &= b_3.
   \end{align*}
   \]
   
   (a) For what values of $b_1$, $b_2$, and $b_3$ does this system have a solution?
   
   (b) Find all solutions to this system for $b_1 = 0$, $b_2 = 1$, and $b_3 = 1$.

3. (10 pts: 5 each) Calculate $e^A$ where
   \[
   A = \begin{bmatrix}
   2 & 0 & 1 \\
   0 & -1 & 0 \\
   1 & 0 & 2 
   \end{bmatrix}
   \]

4. (10 pts: 5 each)
   
   (a) Let $W$ denote the vector space of all functions of $x$ which can be differentiated infinitely many times. Explain why differentiation, $\frac{d}{dx}: W \to W$, is a linear transformation.
   
   (b) Let $V$ be a subspace of $W$ with the following basis
   \[ \{\sin x, \cos x, \sin 2x, \cos 2x\} \, . \]
   
   Find the matrix representing $\frac{d}{dx} : V \to V$ with respect to the above basis.
5. (15 pts: 3 each) Decide whether each of the statements below are true or false. If true, explain why. If false, provide a counterexample or correct the statement.

(a) The linear span in $\mathbb{R}^3$ of \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \) consists of two straight lines.

(b) $A$ is a given $m \times n$ matrix. Define $S$ to be the set of all $\vec{b} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{b}$ has a solution $\vec{x} \in \mathbb{R}^n$. In notation, $$S = \{ \vec{b} \in \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ is solvable} \}$$

Then $S$ is a vector subspace of $\mathbb{R}^m$.

(c) If $A$ is an $n \times n$ diagonalizable matrix, then $0$ cannot be an eigenvalue of $A$.

(d) If $Q$ is an orthogonal matrix, then $\det Q = 1$.

(e) Let $V$ be a vector subspace of $\mathbb{R}^n$ and $V^\perp$ denotes its orthogonal complement. Then every vector $\vec{x} \in \mathbb{R}^n$ must either belong to $V$ or $V^\perp$. That is, either $\vec{x} \in V$ or $\vec{x} \in V^\perp$.

6. (16 pts: 4 each) Let $A$ be an $m \times n$ matrix with rank $r$. List all that you can say about

(i) the values of $m$, $n$, and $r$, and

(ii) the columns of $A$

in each of the following cases.

(a) Depending on $\vec{b}$, $A\vec{x} = \vec{b}$ either has no solutions or only 1 solution.

(b) For any $\vec{b}$, $A\vec{x} = \vec{b}$ always has infinitely many solutions.

(c) Depending on $\vec{b}$, $A\vec{x} = \vec{b}$ either has no solutions or infinitely many solutions.

(d) For any $\vec{b}$, $A\vec{x} = \vec{b}$ always has exactly 1 solution.

7. (15 pts: 5 each) Let $\vec{x}, \vec{y}$ be nonzero vectors in $\mathbb{R}^n$, $n \geq 2$, and let $A = \vec{x}\vec{y}^T$. Show that

(a) $\lambda = 0$ is an eigenvalue of $A$ with $n - 1$ linearly independent eigenvectors and consequently has multiplicity at least $n - 1$.

(b) The remaining eigenvalue of $A$ is

$$\lambda_n = \text{tr } A = \vec{x}^T \vec{y}$$

and $\vec{x}$ is an eigenvector belonging to $\lambda_n$.

(c) If $\lambda_n = \vec{x}^T \vec{y} \neq 0$, then $A$ is diagonalizable.

8. (9 pts: 5, 4) Let $\vec{u}$ be a real vector with length 1, i.e., $\vec{u}^T \vec{u} = 1$.

(a) Show that $P = \vec{u}\vec{u}^T$ is a projection matrix. Recall that a projection matrix is one that satisfies $P^2 = P$ and $P^T = P$.

(b) Show that $T = I - 2P = I - 2\vec{u}\vec{u}^T$ is both symmetric and orthogonal.