Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, January 2020

Instructions: This examination lasts 4 hours. Each problem is worth 15 points. While there are 10 problems, your total score will be calculated by adding up your 8 highest scores. Hence, the maximum total score is $8 \times 15 = 120$ points. Show explicitly the steps and calculations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded for relevant work.

1. Suppose that a 3×3 matrix A satisfies

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}. \tag{1}$$

   (a) Explain why A can be uniquely determined by (1).
   (b) Determine the matrix A.

2. Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 7 \\ -1 & -5 \\ -2 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}. \tag{2}$$

   (a) Show that the system does not have a solution.
   (b) Define the least-squares solution $\mathbf{x}^*$ using the 2-norm (or Euclidean norm) $\|\cdot\|_2$.
   (c) Compute $\mathbf{x}^*$.

3. (a) Find the eigenvalues and associated eigenvectors $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, and $(\lambda_3, \mathbf{v}_3)$ of B such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms an orthonormal basis for $\mathbb{R}^3$, where

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{3}$$

   (b) Using the result of (a), diagonalize B.
   (c) Using the result of (b), show that $B^{2020} = 3^{2019}B$.

4. (a) Find the following decomposition:

$$A = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 10 & 5 \\ 2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & 0 \\ r_{12} & r_{22} & 0 \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = R^T R, \tag{4}$$

   where $r_{11} > 0$, $r_{22} > 0$, and $r_{33} > 0$.
   (b) Show that $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

5. Suppose that the matrix A has the following decomposition:

$$A = \begin{bmatrix} -0.22 & -0.79 & 0.42 & 0.40 \\ -0.28 & 0.50 & 0.82 & -0.02 \\ -0.50 & -0.28 & -0.02 & -0.82 \\ -0.79 & 0.22 & -0.40 & 0.42 \end{bmatrix} \begin{bmatrix} 6.44 & 0 & 0 \\ 0 & 2.55 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.97 & -0.10 & -0.23 \\ 0.18 & -0.93 & -0.33 \\ -0.18 & -0.37 & 0.91 \end{bmatrix}. \tag{5}$$

Briefly answer the following questions:

(a) What are the size and rank of A?

(b) What are the eigenvalues of A^TA and AA^T?

(c) Write down a basis for the null space of A.

(d) Write down a basis for the null space of A^T.

(e) Write down a basis for the range of A.

(f) Write down a basis for the range of A^T.

6. Denote the set of all polynomial functions of degree 2 or less by $\mathcal{P}_2$.

(a) Briefly explain why $\mathcal{P}_2$ is a linear space.

(b) For the inner product

$$(p, q) = \int_{-1}^{1} p(x)q(x)dx, \tag{6}$$

show that $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}x$ are orthonormal.

(c) By performing the Gram–Schmidt process starting from the basis $\{p_0(x), p_1(x), x^2\}$, find an orthonormal basis of $\mathcal{P}_2$.

7. (a) Show that the following identity holds for $k = 1, 2, \ldots$:

$$\left(I + B + B^2 + \cdots + B^{k-1}\right)(I - B) = I - B^k. \tag{7}$$

(b) Assuming its convergence, compute the geometric series $\sum_{k=0}^{\infty} B^k = I + B + B^2 + \cdots$ for

$$B = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}. \tag{8}$$

(c) State when the geometric series of a matrix converges.

8. Let $A, B \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.

(a) Define the matrix p-norm $\|A\|_p$ induced by the vector p-norm $\|\mathbf{x}\|_p$.

(b) Using the definition you stated in (a), prove that $\|AB\|_p \le \|A\|_p\|B\|_p$.

(c) Show that $\|A\|_p\|A^{-1}\|_p \ge 1$.

9. Consider the following iteration:

$$\mathbf{x}^{(k+1)} = \begin{bmatrix} 0 & -\frac{1}{3} \\ -\frac{2}{5} & 0 \end{bmatrix} \mathbf{x}^{(k)} + \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{5} \end{bmatrix}, \quad k = 0, 1, 2, \dots. \tag{9}$$

   (a) Assuming that the iteration converges, compute the convergent solution $\mathbf{x}^*$.

   (b) Derive a recurrence relation for the error vector $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^*$.

   (c) Explain why this iteration converges.

10. You are a TA for a linear algebra class. An exam question asks students to solve a specific linear system $A\mathbf{x} = \mathbf{b}$. The answer key given by your instructor says that all solutions to the system should have the form

$$\mathbf{x} = \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}. \tag{10}$$

   (a) A student wrote that all solutions to the system are

$$\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{with } s \text{ and } t \text{ being any real numbers.} \tag{11}$$

   Should you deduct points from this student for giving an incorrect answer? Explain why or why not.

   (b) Write down all properties of matrix A that you can deduce, such as its size, rank, the dimensions of the four fundamental subspaces, etc.