Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, January 2021

Instructions: This examination lasts 4 hours. Each problem is worth 20 points. While there are 9 problems, your total score will be calculated by adding up your 6 highest scores. Hence, the maximum total score is $6 \times 20 = 120$ points. Show explicitly the steps and calculations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded for relevant work.

1. For the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tag{1}$$
consider its fundamental subspaces, that is, the null spaces of A and A^T and the range (or column) spaces of A and A^T. For each vector space, find a basis. *Note:* You do not need to find orthogonal or orthonormal bases.

2. Assume that a time-dependent matrix $A(t)$ (i.e. each component of A is a function of t) is invertible and differentiable for all t.
   (a) Show that
$$\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}. \tag{2}$$
   Hint: $A(t)A^{-1}(t) = I$.
   (b) By explicitly calculating both sides of (2), show that (2) indeed holds for
$$A(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \tag{3}$$

3. Consider the following linear combination:
$$(1, 2, 3) = c_1(1, 1, 1) + c_2(1, 1, 0). \tag{4}$$
   (a) Show that there do not exist c_1 and c_2 that satisfy (4).
   (b) Find optimal values of c_1 and c_2. Explain in which sense these values are optimal.

4. Consider the following matrix
$$B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5}$$
   (a) Calculate the matrix norms $\|B\|_1$ and $\|B\|_\infty$.
   (b) Calculate the matrix norm $\|B\|_2$.
   (c) Calculate the spectral radius $\rho(B)$ of the matrix B.
   (d) Compute the geometric series
$$\sum_{k=0}^{\infty} B^k = I + B + B^2 + \cdots, \tag{6}$$
   if the series converges. Otherwise, explain why it does not converge.

5. Let S be the subspace spanned by $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (1, 2, 1)$.

   (a) Find an orthonormal basis for S.

   (b) Find the projection matrix $P \in \mathbb{R}^{3\times3}$ that projects a vector in $\mathbb{R}^3$ onto the subspace S.

   (c) Find the projection matrix $Q \in \mathbb{R}^{3\times3}$ that projects a vector in $\mathbb{R}^3$ onto the orthogonal complement $S^{\perp}$ of the subspace S.

6. Diagonalize the following matrix A using an *orthogonal* matrix P:

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \tag{7}$$

7. Compute the singular value decomposition of the following matrix A:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \tag{8}$$

8. Let Null(A) be the null space of A (i.e. the solution space of $A\mathbf{x} = 0$). For each of the following cases, find an example of $A \in \mathbb{R}^{4\times4}$. *Note:* Write down a *specific* matrix for each case.

   (a) $\dim(\mathrm{Null}(A)) = 0$.

   (b) $\dim(\mathrm{Null}(A)) = 1$.

   (c) $\dim(\mathrm{Null}(A)) = 2$.

   (d) $\dim(\mathrm{Null}(A)) = 3$.

   (e) $\dim(\mathrm{Null}(A)) = 4$.

9. Determine whether each of the following statements is true or false. Justify your answers.

   (a) There exist a pair of square matrices A and B such that $AB = I$ but $BA \neq I$.

   (b) For n-dimensional (column) vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}^T\mathbf{y} = 0$ if and only if $\mathbf{x}\mathbf{y}^T = 0$.

   (c) For a nonzero vector $\mathbf{b}$, the solution space of $A\mathbf{x} = \mathbf{b}$ is a vector space.

   (d) The space of 3×3 symmetric matrices has dimension 3.

   (e) If the 1-norm of a matrix B is greater than 1, the series $I + B + B^2 + \cdots$ diverges.