Directions: You have four hours to complete this exam. One crib sheet is allowed. Credit will be awarded mainly based on the level of work and explanation (no explanation = no credit).

1. (12 points – 3 points each) Answer each of the following unrelated problems.

   (a) What can you say about existence and uniqueness of solutions for the following initial-value problem for $y(t)$ on the interval $1 \le t < \infty$: $y'' + p(t)y' + y = \ln(t)$, $y(1) = 0$, $y'(1) = 0$, where
   $$p(t) = \begin{cases} t-2 & 1 \le t \le 2 \\ \ln(t-1) & t > 2 \end{cases}$$

   (b) For the differential equation $\dfrac{dy}{dt} = \cos(y)/(y+1)$, classify the stability of the equilibrium point at $y = \pi/2$.

   (c) The following differential equation represents a system that is (i) under-damped, (ii) critically damped, or (iii) over-damped?
   $$y'' + 2\sqrt{2}y' + 2y = 0$$

   (d) Sketch the phase-plane ($y'(t)$ vs. $y(t)$) trajectory for the differential equation $y''+4y = 0$, with $y(0) = 1$, $y'(0) = 0$.

2. (6 points) Classify, to the best of your abilities, the following differential equations for $y(t)$ (do not solve):

   (a) $\dfrac{d^3y}{dt^3} + \dfrac{dy}{dt} - \exp(t)y = \exp(-t)$

   (b) $\dfrac{dy}{dt} = \exp(y^2)$

3. (12 points) Find the solutions to the following initial-value problems:

   (a) $\dfrac{dy}{dt} = y\cos(t), \quad y(\pi) = 2$

   (b) $w\exp(2xw)+x+x\exp(2xw)\dfrac{dw}{dx} = 0$, $w(1) = 0$ (implicit solution for $w(x)$ acceptable)

4. (12 points) Consider the following differential equation for $y(t)$:
   $$\frac{d^2y}{dt^2} - y = t^2$$

   (a) Find two linearly independent solutions to the homogeneous problem. How do you know they are linearly independent?

   (b) Determine the general solution $y(t)$ for the non-homogeneous problem.

5. (10 points) Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ for $\mathbf{x}^T(t) = [x_1(t), x_2(t)]$ where A is defined as

$$A = \begin{bmatrix} -2 & 1 \\ 4 & 1 \end{bmatrix}.$$

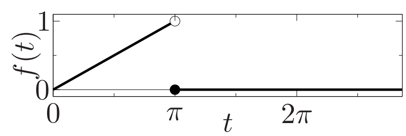
Determine the general solution and classify the stability of the equilibrium point $\mathbf{x} = \mathbf{0}$.

6. (10 points) Determine the linearly independent eigenfunctions of the following boundary-value problem:

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$$

7. (10 points) Determine the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$ of the solution of the following initial value problem. You don't need to solve for $y(t)$.

$$y'' - 5y' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0$$



8. (10 points) The Oster double-bucket method is used to produce solutions with time varying salinity. A typical set-up consists of two 1000ml containers (A and B) connected by a pipe in which the flow rate from A to B is constant at 10ml/min. Initially, pure water is placed in container A while a saline solution with concentration of 0.05g/ml is placed in container B. Container B is continually mixed and a pipe empties the solution from container B at an undetermined constant rate R. Using appropriate first order differential equations and initial conditions describing the solution volume and salt mass in Container B, determine an expression for the salt mass as a function of time. Then determine the flow rate R required so that the salt concentration in container B decreases linearly in time.

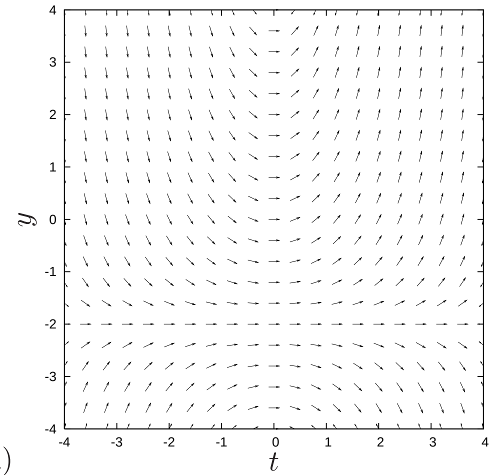
9. (10 points) For the following differential equation, find the first four nonzero terms in each of two linearly independent power-series solutions about the origin:

$$y'' + \sin(x)y = 0$$

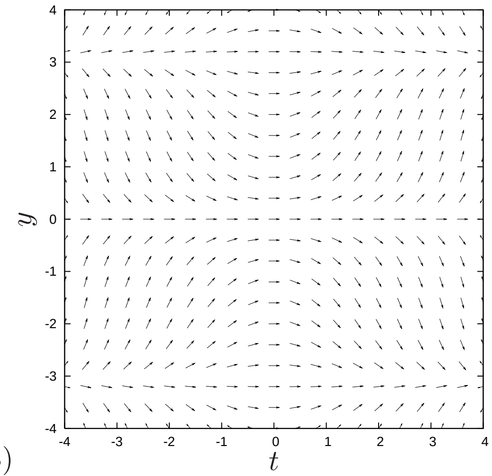
10. (8 points) For each of the following two differential equations, identify the appropriate direction field (A) – (D).

(1) $\dfrac{dy}{dt} = \dfrac{1}{2}ty + t$

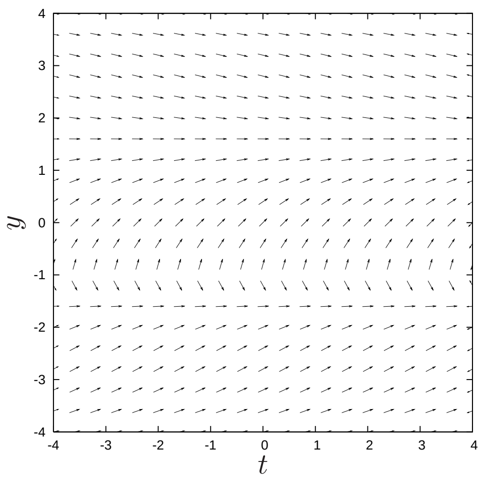
(2) $\dfrac{dy}{dt} = y \exp\left(-\dfrac{t^2}{2}\right),$



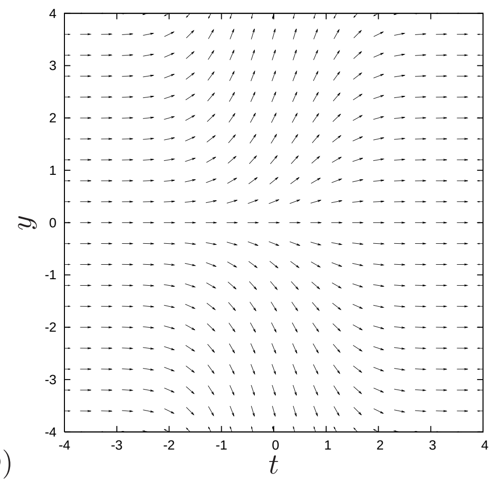
(A)



(B)



(C)



(D)