Instructions: Credit will awarded mainly based on the **quality of your work** and the **clarity of your explanations**.

1. (18 points, 6 points each) Answer each of the following questions.

   (a) Solve the IVP
   $$y' = \cos^2(y), \quad y(0) = \frac{\pi}{2} \, .$$

   (b) Find the general solution of the IVP
   $$y' - \tan(x)y = \tan(x) \, .$$

   (c) Solve the integral equation
   $$y^2(x) = \int_0^x sy(s)\, ds \, .$$

2. (12 points) Consider the ODE
   $$\dot{y} = y^2(1 - y^4) \, .$$

   (a) (3 points) What are the equilibrium points?

   (b) (6 points) Determine the stability of the equilibrium points.

   (c) (2 points) What is $\lim_{t\to\infty} y(t)$ if $y(0) = 0.5$?

3. (15 points, 5 points each) Consider the ODE
   $$y'' + \omega_0^2 y = \cos(\omega_1 t) \, , \quad \omega_0, \omega_1 \in \mathbb{R}.$$

   (a) Find the general solution when $\omega_1 \neq \omega_0$.

   (b) Describe the change in the behavior of the solution as $\omega_1 \to \omega_0$.

   (c) Explain qualitatively how your answer to part (b) would change for the equation $y'' - \omega_0^2 y = \cos(\omega_1 t)$.

4. (9 points) Consider the eigenvalue problem

$$y'' + \lambda y = 0 \ , \quad y'(0) = 0, \ y(\pi) = 0 \ .$$

   (a) (6 points) Find the eigenvalues and corresponding eigenfunctions.

   (b) (3 points) Write down the condition on $f(x)$ such that

   $$y'' + \lambda y = f(x) \ , \quad y'(0) = 0, \ y(\pi) = 0$$

   has a <u>bounded</u> solution.

5. (9 points) Consider the ODE

$$y'' + y - y^3 = 0 \ .$$

   (a) (3 points) Find the equilibrium solutions.

   (b) (6 points) Linearize the ODE around one of the equilibrium solutions of your choice and determine its stability.

6. (6 points) Find the power series $y = \sum_{t=0}^{\infty} a_n t^n$ for the solution of

$$y'' + y = 0 \ , \quad y(0) = A, \ y'(0) = B$$

where A and B are constants.

GOOD LUCK!