Applied Mathematics ODE Preliminary Exam

Instructions. Solve each of the problems below. Credit will awarded mainly based on the quality of your work and the clarity of your explanations.

1. Solve the initial value problem

$$t\frac{dy}{dt} + 2y = e^{-2t} \tag{1}$$
$$y(1/2) = 0. \tag{2}$$

   On what interval are we guaranteed to have a unique solution?

2. Consider the equation:

$$y''(t) + ay'(t) + by(t) = 0 \tag{3}$$

   where $a < 0$ and $b < 0$. Find the the general solution. What is the behavior as $t \to \infty$?

3. Given the differential equation

$$\frac{dy}{dt} = ay - y^3 \tag{4}$$

   (a) Find the critical (equilibrium) points.

   (b) Determine the stability of these critical points (asymptotically stable, unstable, or semistable) and draw the phase line.

   (c) Draw the bifurcation diagram for this equation.

4. A tank in the lab has capacity of 100 gallons. Initially, the tank has 100 gallons of water with 10 lbs of salt mixed in solution. Water containing 1 lb of salt per gallon flows into the tank at a rate of $2 + \cos(t)$ gallons per minute. The mixture in the tank flows out at the same rate (after being well mixed).

   (a) Give an initial value problem for the rate of change of salt in the tank.

   (b) Solve for the amount of salt in the tank at any time t.

   (c) Find the limiting amount of salt in the tank as $t \to \infty$.

5. Find the general solution and draw the phase portrait for the system of equations,

$$\vec{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \vec{x} \tag{5}$$

6. Show that the differential equation,

$$y'' - 2xy' - 4y = 0,$$

possesses two linearly independent solutions that are analytic at the origin. Evaluate them by the power series method.

7. Solve the differential equation,

$$x^2 y'' + \frac{x}{2} y' - 2x^2 y = 0,$$

about $x = 0$ using the method of Frobenius.

8. Determine the eigenvalues and eigenfunctions of the differential equation,

$$y'' + 2y' + (1+\lambda)y = 0, \quad \text{in } 0 < x < \pi$$
$$y'(0) = y'(\pi) = 0.$$