Applied Mathematics, Ordinary Differential Equations
Preliminary Exam—May 2019
May 15, 2019 9am-1pm
Instructions

Read the following instructions carefully:

- Write each problem on a separate page.
- Write the number of the problem, and your name, in the top right corner of your answer sheet.
- It is important to show your work for each problem. Credit will NOT be given for correct answers without justification. Also, partial credit will be given for incorrect answers if some of the work is correct.
- Clearly mark out (cross out) any work that you are not including in your answer and you do not want graded.
- Be sure to staple your exam at the end and hand it in.
- Good luck!

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1. (**5 Points**)

   (a) Prove that the following problem has a unique solution

   $$\dot{x}(t) = ix(t), \qquad x(0) = 1.$$

   (b) Prove Euler's formula

   $$e^{it} = \cos(t) + i\sin(t).$$

2. (**10 Points**) Consider the ecological model

   $$\begin{aligned} \dot{x} &= x(1 - x + \alpha y) \\ \dot{y} &= y(1 - 2y + \beta x), \end{aligned}$$

   where x and y represent the population of two distinct species, assumed to be equal to x_0 and y_0 at $t = 0$.

   (a) Describe the interactions between the two species in the following cases:
      i. $\alpha < 0$ and $\beta > 0$
      ii. $\alpha < 0$ and $\beta < 0$
      iii. $\alpha > 0$ and $\beta > 0$.

   (b) Reformulate the above problem in the form

   $$X = F(X), \qquad X(0) = X_0$$

   and compute the Jacobian of F.

   (c) Prove that for any α and β, the above system has a unique solution.

   (d) For what follows, consider the case $\alpha = \beta = 1$. Find the stationary points and discuss their stability.

   (e) Sketch the phase diagram of the linearized problem around each stationary point.

   (f) Explain why you can sketch the phase diagram of the full system from the ones of the linearized problem and sketch it.

   (g) Describe the asymptotic behavior of the solution and compare it to your answer. Is it consistent with your interpretation in a) iii. above?

3. (**10 Points**) Consider a string of length 1 in a two-dimensional space (xy plane) tied between the points $x = 0$ and $x = 1$ and free to move at any point $x \in (0, 1)$. Denote $h(x, t)$ its local height (in the y-direction) at (x, t). The dynamics of this string is governed by the vibrating string equation

   $$\frac{\partial^2 h(x,t)}{\partial t^2} - c^2 \frac{\partial^2 h(x,t)}{\partial x^2} = 0, \tag{1}$$

   where c is the wave speed. The initial and boundary conditions are

   $$h(x, 0) = h_0(x), \qquad h(0) = h(1) = 0,$$

   respectively. We are interested in finding the solution of the above PDE using ODE theory only.

(a) We seek general solutions of the form

$$h(x,t) = f(x)g(t).$$

Prove that for any such solution, the ratios $\frac{f''(x)}{f(x)}$ and $\frac{g''(t)}{c^2 g(t)}$ are equal to the same real constant $\lambda \in \mathbb{R}$. We will assume that $\lambda \neq 0$.

(b) Write two second-order ODEs for $f(x)$ and $g(t)$ involving λ. What are the initial condition for $g(t)$ and boundary conditions for $f(x)$?

(c) Consider the case $\lambda > 0$. Find the general form of $f(x)$. Apply the boundary conditions and show that the only possible solution is $f(x) = 0$.

(d) For what follows, assume $\lambda < 0$. Find the general form of $f(x)$. Apply the boundary conditions and show that there exist a non-zero $f(x)$ if and only if $\lambda = n^2\pi^2$ where $n \in \mathbb{R}$.

(e) Let $\lambda_n = n^2\pi^2$. For each λ_n write down a generate solution of the PDE equation (1) in the form $h_n(x,t) = f_n(x)g_n(t)$ (do not worry about the initial condition). Make sure to show the expressions for $f_n(x)$ are $g_n(t)$.

(f) Prove that $\forall \alpha_n,\ n = 1, \ldots, N$

$$h(x,t) = \sum_{n=1}^{N} \alpha_n h_n(x,t),$$

satisfies the vibrating string equation.

(g) Can we conclude that we have found the general form of the solution to the above PDE problem? Explain why.

4. (**7 Points**) Consider the differential equation below for the function $y(x)$:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$$

(a) Use the following transformation of the independent variable $x = e^t$ to arrive to the following differential equation in $y(t)$

$$y'' + y = e^t. \tag{2}$$

(b) Give a particular solution to the resulting equation in terms of $y(t)$ and in terms of $y(x)$.

(c) Give the solution to the homogeneous version of this equation, in terms of $y(x)$.

(d) Give the general solution to this equation.

5. (**8 Points**) Suppose a nonlinear system for the functions $z_x(t)$ and $z_y(t)$ has 4 stationary points: A:$(0,0)$, B:$(1,0)$, C:$(0,2)$, and D:$(4,3)$. Their respective eigenvalues and eigenvectors are given below. Sketch its phase portrait.

A: $\lambda_1 = 1$, $\boldsymbol{v}_1 = (0,1)$, $\lambda_2 = 3$, $\boldsymbol{v}_2 = (1,0)$.

B: $\lambda_1 = -2$, $\boldsymbol{v}_1 = (1,0)$, $\lambda_2 = 1$, $\boldsymbol{v}_2 = (1,1)$.

C: $\lambda_1 = -1$, $\boldsymbol{v}_1 = (0,1)$, $\lambda_2 = 2$, $\boldsymbol{v}_2 = (2,1)$.

D: $\lambda_1 = -1 + 3i$, $\boldsymbol{v}_1 = (1,i)$, $\lambda_2 = -1 - 3i$, $\boldsymbol{v}_2 = (1,-i)$, and at $z_x = 4$, $z_y = 3.001$, $z_x' > 0$.

6. (**10 Points**) Consider the following system of ordinary differetial equations

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x + \begin{bmatrix} e^t \\ t \end{bmatrix}. \tag{3}$$

   (a) Use the method of undertermined coefficients to find a particular solution for (3).

   (b) Use the method of variation of parameters to find the general solution for (3).

7. (**10 Points**) Consider the heat equation with steady state source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x) \tag{4}$$

subject to the boundary and initial conditions:

$$u(0,t) = 0, u(L,t) = 0, \text{ and } u(x,0) = f(x). \tag{5}$$

   (a) Obtain the solution to the above initial bopundary value problem by the method of eigenfunction expansion. *Hint: Consider the eigenfunctions of the related homogeneous problem. Use separation of variables to arrive to an x-dependent eigenvalue problem. Use the eigenfunctions you find to expand the unknown solution $u(x,t)$ in a series of the eigenfunctions (similarly you will express the right hand side $Q(x)$ as an eigenfunction expansion). Finally, plug in this series solution to (4) and use term-by-term differentiation to arrive to a nonhomogenous linear first-order ordinary differential equation, which you can solve by the method of integrating factor (multiply by $e^{\lambda_n kt}$, where λ_n are the eigenvalues).*

   (b) Show that the solution approaches a steady state solution.