Applied Mathematics
ODE Preliminary Exam
January 14, 2020 9am-1pm

Instructions and advice:

- For each problem, read carefully every single question. Then, read carefully every single question again.
- Within each problem, questions are often largely independent. Not answering a question will not necessarily prevent you from answering the following ones.
- Make sure to justify all your answers.
- Good luck !

1. In this problem, we are interested in estimating the cooking time of a chocolate brownie. To do so we first consider a 1D brownie of size L placed in an oven and monitor its temperature $T(x,t)$. We assume that the extremities of the brownie are kept at a constant temperature;

$$T(0,t) = T(L,t) = T_1$$

and the initial temperature profile is

$$T(x,0) = T_0(x,t)$$

The evolution of the temperature is given by the heat equation

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}, \tag{1}$$

where α is the heat diffusivity of the brownie ($\alpha > 0$).

(a) What are the PDE, iniitial condition and boundary condition satisfied by the rescaled temperature $u(x,t) = T(x,t) - T_1$?

(b) We seek general solutions of the form

$$u(x,t) = f(x)g(t).$$

Prove that for any such solution, the ratios $\frac{f''(x)}{f(x)}$ and $\frac{g'(t)}{\alpha g(t)}$ are equal to the same real constant $\lambda \in \mathbb{R}$. We will assume that $\lambda \neq 0$.

(c) Write the ODEs for $f(x)$ and $g(t)$ involving λ. What are the initial condition for $g(t)$ and boundary conditions for $f(x)$?

(d) Why did we choose to call this constant λ ?

(e) Consider the case $\lambda > 0$. Find the general form of $g(t)$. Explain why the solution is non-physical.

(f) For what follows, assume $\lambda < 0$. Find the general form of $f(x)$. Apply the boundary conditions and find under what condition there exist a non-zero solution $f(x)$

(g) For each $\lambda_n = -\frac{n^2\pi^2}{L^2}$ $h_n(x,t) = f_n(x)g_n(t)$ (do not worry about the initial condition). Make sure to show the expressions for $f_n(x)$ are $g_n(t)$.

(h) Prove that $\forall \alpha_n,\ n = 1, \dots, N$

$$u_N(x,t) = \sum_{n=1}^{N} \alpha_n h_n(x,t),$$

satisfies the PDE and the boundary conditions you determined in (a).

(i) Can you prove that for all sequence α_n the serie $u_\infty(x,t) = \sum_{n=1}^{\infty} \alpha_n h_n(x,t)$ satisfied the PDE and the boundary conditions ?

(j) How can you estimate the cooking time ? (hint: for our recipe the brownie is cooked when the temperature is uniform)

(k) Considering a 2D brownie of size $L_1 \times L_2$, what would be the cooking time ?

(l) What about a N-dimensional brownie of size $L_1 \times .. \times L_N$?

2. Solve the following equations.

   (a) $y' - y = 2te^{3t}$ with $y(0) = 1$

   (b) $y' = cos(x)/(1 + e^y)$ with $y(0) = 1$

   (c) $2y'' + 3y' + y = x^2$ with $y(0) = 1$ and $y'(0) = 0$

   (d) $y''' + xy = 0$ with $y(0) = 1$, $y'(0) = 0$ and $y''(0) = 0$

3. (a) Assuming that what you know for real power series holds true for complex variables, prove Euler's formula

$$e^{it} = \cos(t) + i\sin(t) \qquad \forall t \in \mathbb{C}.$$

   (b) Prove that

$$\sin t = \frac{e^{it} - e^{-it}}{2i} \qquad \forall t \in \mathbb{C}. \tag{2}$$

   (c) Solve the following equation for x

$$\sin(x) = 2. \tag{3}$$

4. (a) Describe the asymptotic behavior (in the limit $t \to \infty$) of the following 8th order ODE ? Make sure to justify your answer

$$y^{(8)} - 2y^{(4)} + 1 = 0. \tag{4}$$

   (b) Same question for the general $2n - th$ order ODE

$$y^{(2n)} - 2y^{(n)} + 1 = 0. \tag{5}$$

5. The sun emits lights by fusing hydrogen atoms in its core. The most sophisticated physical models predict that it will stop doing so in about 5 billion years (5×10^9 years) when its hydrogen reserves will be exhausted.

   The goal of this problem is to create our own estimation of the time t_f when the sun will stop emitting light (i.e when its hydrogen reserve will have been depleted) using simple ODE models. We will denote $N(t)$ the total number of hydrogen atoms, and take for the initial condition

$$N(0) = N_0 = 10^{57} \tag{6}$$

which is our estimation of the current reserve of hydrogen in the sun. The time t will be in years.

(a) As a first approach we will consider that the hydrogen reserve is model by the ODE

$$\dot{N} = -\alpha N$$

where $\alpha = 1.17 \times 10^{-11} year^{-1}$ is the rate of fusion.

i. Solve the ODE and find the solution $N(t)$.

ii. We define t_f as the time when the hydrogen reserves have diminished by 99%. Find the expression of t_f in term of the problem parameters and compute an approximated numerical value (Hint: $\log(0.01) \approx -4.6$).

iii. How good is this first approximation ?

(b) As atoms are fusing, the cores becomes denser, the gravitational pressure increase and it becomes easier for atoms to fuse. In other word, the rate of fusion should increase with the number of atoms already fused $(N_0 - N(t))$ and the sun should be brighter and brighter . We therefore consider a second more sophisticated model

$$\dot{N} = -\alpha \left(1 + \epsilon \frac{N_0 - N}{N_0}\right) N = -\alpha \left((1 + \epsilon) - \epsilon \frac{N}{N_0}\right) N, \tag{7}$$

where the coefficient ϵ can be seen the "brightness increase". Based on the information found on wikipedia we will take $\epsilon = 100$.

i. We define $x = \frac{N}{N_0}$ and $\eta = 1 + \epsilon$. Using the initial condition 6 and the above ODE 7 find a problem (i.e ODE and an initial condition) of which $x(t)$ is the solution.

ii. Solve the problem and find x(t).

iii. Show that the final time t_f is

$$t_f = \frac{1}{\alpha\eta} \log\left(\frac{\eta}{0.01} - \epsilon\right)$$

iv. Compute a (rough) approximation of t_f (hint: $\log(10^4) \approx 9.2$). How good is this second approximation ?