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By completing this assignment, you acknowledge and confirm that you will not give or receive any unauthorized assistance on this assignment/examination. You will conduct yourself within the guidelines of the university academic integrity guidelines.

Applied Mathematics, Ordinary Differential Equations
Preliminary Exam—January 2021
January 13, 2021 9am-1pm

Instructions and advice:

- For each problem, read carefully every single question. Then, read carefully every single question again.
- Within each problem, questions are often largely independent. Not answering a question will not necessarily prevent you from answering the following ones.
- Make sure to justify all your answers.
- Good luck!

Problem 1: Qualitative Analysis

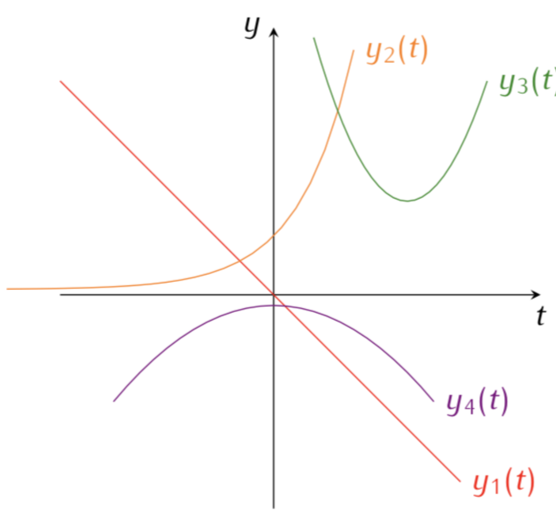

Figure 1

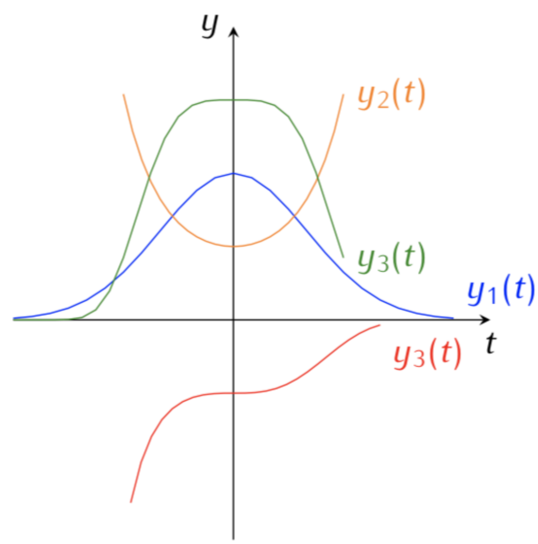

Figure 2

1. We consider the following ordinary differential equation

$$y' = \frac{e^t}{t^2+1} y. \tag{1}$$

   (a) Among the four functions $y_{i-1..4}$ plotted on figure 1, determine which ones are likely to be a solution of the above equation and which ones are unlikely to.

   (b) Compute the second-order derivative of the solution y'' as a function of y. What can you say about the convexity of the solution?

2. For each of the following ODE

(a) $y' = -ty$

(b) $y' = ty$

(c) $y' = -t^2 y$

(d) $y' = -t^3 y$

one of its solution is plotted on figure 2. Identify which solution correspond to which equation.

Problem 2: Existence, Uniqueness, and their Implications

We consider the general Initial Value Problem (IVP)

$$y' = f(y,t), \quad (2)$$
$$y(0) = y_0. \quad (3)$$

1. What conditions the function $f(y,t)$ must satisfy for the above problem to admit a unique solution?

2. We assume that these conditions are satisfied and that a unique solution solution is always for all times. We consider two solutions of the above problem y_1, y_2 for two initial values y_0^1 and y_0^2. Prove that if there exist a time t_c such that $y_1(t_c) = y_2(t_c)$, then $y_1(t) = y_2(t)$ for all time and thus $y_0^1 = y_0^2$.

3. From now on, we assume that the function $f(y,t)$ has the following form

$$f(y,t) = e^{4-y^2} - 1. \quad (4)$$

   (a) Does the above function satisfy the conditions you enunciated in 2) a)? We will admit for the rest of the problem that if a unique solution exists, it is defined for all times.

   (b) Find two constant solutions (for two distinct initial conditions) to the IVP. We will call these constant α and β with $\alpha < \beta$.

4. We are now interested in the particular solution for the initial condition

$$y(0) = 0. \quad (5)$$

   (a) Using the result from 2) b) prove that solution $y(t)$ for the above initial condition cannot intersect the constant solutions you found in 2) c) ii).

   (b) Conclude that for all time $\alpha < y(t) < \beta$

   (c) What is the limit of $y(t)$ as t goes to $\pm\infty$?

   (d) Discuss the convexity of y(t).

   (e) Sketch the solution.

Problem 3: Power Series

Consider the Initial value problem

$$y' = y^2, \quad (6)$$
$$y(0) = 1. \quad (7)$$

1. Discuss wether it is a good strategy to seek for a power series solution.
2. Prove that the unique solution is

$$y(t) = \frac{1}{1-t}$$

3. Using a Taylor expansion, show that the power series expansion $p(t)$ of $y(t)$ around $t = 0$ is

$$p(t) = \sum_{n=0}^{\infty} t^n.$$

4. What is the radius of convergence of the above power series? For which t are the power serie and the solution identical?
5. Are the function and its power series well defined at $x = 1$ and $x = -1$?
6. Are you answer to 4) and 5) consistent?

Problem 4: First-Order ODE

We consider $y(t)$ the general solution of the following Bernoulli equation

$$2y' + y = (t-1)y^3 \tag{8}$$

and introduce the variable $z(t) = y(t)^{-2}$

1. Express z' as a function of y'.
2. Prove that z is the general solution of

$$a(t)y' + b(t)y = g(t), \tag{9}$$

   and determine the functions a, b and g.
3. What are the homogeneous and general solutions of (9)?
4. What is the general solution of the Bernoulli's equation?

Problem 5: System of ODE

Consider the system of coupled ODE

$$X' = AX + B(t) \tag{10}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B(t) = \begin{pmatrix} 4e^t \\ 0 \end{pmatrix}$$

1. Does the above system have a unique solution?

2. Does it have stationary points? If so discuss their stability.
3. What are the eigenvalues of A ?
4. What is the general solution $X_H(t)$ of the homogeneous system?
5. What is the asymptotic behavior of $X_H(t)$ as $t \to \infty$? In particular, specify whether $X_H(t)$ converges or not.
6. Verify that

$$X_P(t) = \begin{pmatrix} e^t - e^{-t} + 2te^t \\ e^{-t} - e^t + 2te^t \end{pmatrix}$$

is a particular solution of the inhomogeneous system 10.