# A conjectured hierarchy of length scales in a generalization of the Navier–Stokes-alpha equation for turbulent fluid flow

ELIOT FRIED
Department of Mechanical and Aerospace Engineering
Washington University in St. Louis
St. Louis, MO 63130-4899, USA

### Abstract

Fluid flows are often classified as either laminar or turbulent. In a laminar flow, the velocity and pressure fields are smooth, deterministic functions of position and time. In a turbulent flow, these fields exhibit erratic spatial and temporal fluctuations. Roughly, turbulent fluid flow can be thought of as a cascade in which energy is transferred from large to increasingly smaller eddies. Eventually, at the smallest scales, viscosity dissipates the energy into heat. The range of eddy scales present in a turbulent flow typically spans many orders of magnitude: for instance, the eddies present in the boundary layers over the wings of a commercial aircraft can be as small as tens of micrometers while those present in the wake of can be as large as tens of meters. The central challenge of turbulence lies in dealing with the tremendous range of spatial scales.

The broad consensus among fluid mechanicians is that the Navier–Stokes equation embodies all of the features of turbulent fluid flow. Direct numerical simulation (DNS), attacks the Navier–Stokes equation without averaging any of the turbulent eddies. However, for most realistic applications, DNS continues to provide a formidible computational challenge, even with access to state-of-the-art supercomputers. For this reason, there remains a strong interest in alternative methods that resolve only large-scale motions while modeling small-scale motions via filtering. While they reduce computational costs, the additional dissipation associated with filtering can lead to artificially sluggish flows. The recently developed Navier–Stokes-$\alpha$ equation provides a model for which this difficulty appears to be less pronounced.

We present a continuum-mechanical formulation and generalization of the Navier–Stokes-$\alpha$ equation based on a general framework for fluid-dynamical theories with gradient dependencies. That generalization entails two additional problem-dependent length scales $\alpha$ and $\beta$, with $\alpha$ being energetic and $\beta$ being dissipative. When these scales are equal, our flow equation reduces to the Navier–Stokes-$\alpha$ equation. the conventional view of the turbulent energy cascade suggests, however, that $\alpha$ and $\beta$ should characterize the average scales of eddies in the inertial and dissipative subranges and should therefore obey $\beta < \alpha$. In contrast to the original derivation of the Navier–Stokes-$\alpha$ equation, which relied on Lagrangian averaging, our formulation delivers boundary conditions and thermodynamically-based Lyapunov relations. For a confined flow, our boundary conditions involve an additional parameter $\ell$, with $|\ell|$ being a length scale characteristic of the eddies found near walls. Based on a comparison with DNS studies of fully-developed turbulent flow in a rectangular channel of height $2h$, we find that $\alpha/\beta \sim \mathrm{Re}^{0.470}$ and $|\ell|/h \sim \mathrm{Re}^{-0.772}$, where Re is the Reynolds number. The first result indicates that the choice $\alpha = \beta$ required to reduce our flow equation to the Navier–Stokes-$\alpha$ equation is likely to be problematic. The second result is strikingly reminiscent of the classical scaling relation $\eta/L \sim \mathrm{Re}^{-3/4}$ for the ratio of the Kolmogorov microscale $\eta$ to the integral length scale $L$. The numerical data also suggests that $|\ell| \leq \beta$. We are therefore led to conjecture a hierarchy, $|\ell| \leq \beta < \alpha$, involving the three length scales entering the theory.