UC Merced: MATH 21 — Final Exam — 16 May 2006

On the front of your bluebook print (1) your name, (2) your student ID number, (3) your instructor's name (Bianchi) and (4) a grading table. Show all work in your bluebook and BOX IN YOUR FINAL ANSWERS where appropriate. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, calculators and crib sheets are not permitted. There are a total of eight problems and a total of 150 points. Please start each of the eight problems on a new page.

1. (5 points each) Answer the following Always True (T) or False (F). Only your final answer will be graded on these problems.

   (a) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$, then f is continuous at $x = a$.

   True

   (b) $\displaystyle\int \left[2x\cosh(x^2)\left(\frac{x}{2}+\cos(x)\right) + \sinh(x^2)\left(\frac{1}{2}-\sin(x)\right)\right]dx = \sinh(x^2)\left(\frac{x}{2}+\cos(x)\right)+C$

   True

   (c) If f and f' are defined for all x, and f has a maximum at $x = 1$, then $f'(1) = 0$.

   True

   (d) $\displaystyle\int_{-2}^{2} \frac{1}{x^2}dx = \left[\frac{-1}{x}\right]_{-2}^{2}$

   False

   (e) If $\displaystyle\int \frac{1}{(x^2+3x)^3}dx$, then one can make the substitution $u = x^2+3x$ and can rewrite the integral as follows: $\displaystyle\int \frac{1}{u^3}du$.

   False

2. (5 points each) Find all values of x where the given functions are DISCONTINUOUS. Justify your answers.

   (a) $f(x) = \begin{cases} 2x+5, & x<1 \\ x^2-2x+8, & 1 \geq x \geq 3 \\ x^2+1, & 3<x \end{cases}$

   $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+5) = 2(1)+5 = 7$
   $f(1) = (1)^2 - 2(1) + 8 = 1 - 2 + 8 = 7$ Since these agree, f is NOT discontinuous at $x = 1$. If students plug in $x = 1$ into $2x + 5$ without the limit, this is ok.

   $f(3) = (3)^2 - 2(3) + 8 = 9 - 6 + 8 = 11$
   $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2+1) = (3)^2 + 1 = 10$
   Since these do NOT agree, f is discontinuous at $x = 3$.

Note that since there are no other pieces and each piece is a continuous function, we have checked every necessary point.

ANSWER: $x = 3$

(b) $g(x) = \dfrac{2x+5}{2x^2 - 4x - 30}$

Note that g is continuous everywhere except where the denominator is zero:
$2x^2 - 4x - 30 = 0$
$2(x^2 - 2x - 15) = 0$
$x^2 - 2x - 15 = 0$
$(x-5)(x+3) = 0$
$x = 5, x = -3$ both make the denominator zero, so:

ANSWER: $x = -3, 5$

(c) $h(x) = \dfrac{x-2}{\sqrt{x^2+1}}$

Two things could cause problems here:
The denominator being zero.
A negative inside the squareroot.

Note first that for the denominator to be zero, we need $\sqrt{x^2+1} = 0$ or $x^2 + 1 = 0^2 = 0$. This leads us to a change of the second problem. Now the inside cannot be negative or zero.
But we note that $x^2+1 > 0$ for all x, so there are no values of x that make h discontinuous.

ANSWER: NONE

3. (5 points each) A baseball thrown directly upward at 16 ft/sec has velocity $v(t) = 16 - 32t$ ft/sec at time t seconds. The ball is at a height of 32 ft when it is released.

(a) When does the baseball reach the peak of its flight?

At the peak of its flight, the velocity would be zero:
$16 - 32t = 0$
$16 = 32t$
$t = \frac{16}{32} = \frac{1}{2}$ seconds
So the baseball reaches the peak of its flight after half a second.

(b) What is the ball's height at the peak of its flight?

Let height at time t be given by $h(t)$. Then
$h(t) = (t)dt = \int(16 - 32t)dt = 16t - 16t^2 + C$
We know that the initial height is 32 feet, so $h(0) = 32$
$32 = 16(0) - 16(0)^2 + C$ or $C = 32$
So $h(t) = -16t^2 + 16t + 32$
Since the baseball reaches its peak after half a second, we get:
$h(\frac{1}{2}) = -16(\frac{1}{2})^2 + 16(\frac{1}{2}) + 32 = -4 + 8 + 32 = 36$ feet.

(c) When does the ball hit the ground?

When the ball hits the ground, its height is zero, so:

$$
\begin{aligned}
&0 = -16t^2 + 16t + 32 \\
&0 = -16(t^2 - t - 2) \\
&0 = t^2 - t - 2 \\
&0 = (t-2)(t+1) \\
&t = 2, -1
\end{aligned}
$$

The negative answer makes no sense, so the answer must be 2 seconds.

4. (5 points each) Find the following limits, if they exist. Justify your answers. If they do not exist, explain why not.

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x-2}}{e^{2x}}$

If I imagine plugging in infinity or just a REALLY BIG number, I see:

$\displaystyle\lim_{x\to\infty} \frac{\sqrt{x-2}}{e^{2x}} = \frac{\infty}{\infty}$ which is one of the L'Hopital forms, so I take the derivative of the top and of the bottom separately.

$$
\begin{aligned}
&\lim_{x\to\infty} \frac{\sqrt{x-2}}{e^{2x}} = \lim_{x\to\infty} \frac{\frac{1}{2}(x-2)^{-1/2}(1)}{2e^{2x}} \\
&= \lim_{x\to\infty} \frac{1}{4e^{2x}\sqrt{x-2}} \\
&= \frac{1}{\infty} = 0
\end{aligned}
$$

(b) $\displaystyle\lim_{x\to -1^+} \sqrt{x+1}$

Since I only want the right hand limit, there is no problem here.

$$\lim_{x\to -1^+} \sqrt{x+1} = \sqrt{(-1)+1} = \sqrt{0} = 0$$

5. (10 points) Use the definition of derivative along with the Mean-Value Theorem for Definite Integrals to show that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

where a is a constant and f is continuous.

The MVT for Definite Integrals
If g is continuous on $[a, b]$, then at some point c, where $a \le c \le b$,

$$g(c) = \frac{1}{b-a}\int_a^b g(x)dx.$$

$$\frac{d}{dx}\int_a^x f(t)dt = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$

$= \lim_{h\to 0} f(c)$ for some c such that $x \le c \le x+h$ by the MVT for Definite Integrals.
$= f(c)$ for some c such that $x \le c \le x+0$ (taking the limit)
$= f(x)$ since $x \le c \le x$ implies that $c = x$.

6. Find the requested information in the following differentiation problems:

   (a) (10 points) Find values of a and b so that the function $y = axe^{-bx}$ has a local maximum at the point $(2, 10)$.

   First of all, if y has a local maximum at $(2, 10)$, we must get $y = 10$ when we plug in $x = 2$.
   $10 = a(2)e^{-b(2)}$
   $5 = ae^{-2b}$ Let's wait on the rest of this.

   To be a maximum, y' must be zero or undefined.

   $y' = \frac{d}{dx}(ax) \cdot e^{-bx} + (ax) \cdot \frac{d}{dx}(e^{-bx})$
   $= ae^{-bx} - abxe^{-bx}$

   Now that we know y', we see that it is undefined nowhere, so $y'(2) = 0$.

   $0 = ae^{-b(2)} - ab(2)e^{-b(2)}$
   $0 = ae^{-2b} - 2abe^{-2b}$
   $0 = ae^{-2b}(1 - 2b)$

   So either $ae^{-2b} = 0$ or $1 - 2b = 0$. But since the other equations told us that $ae^{-2b} = 5$, it can't be zero, so:

   $1 - 2b = 0$
   $1 = 2b$
   $b = \frac{1}{2}$

   Now we go back to the other equations:

   $5 = ae^{-2b}$ becomes $5 = ae^{-2(1/2)}$
   $5 = ae^{-1}$
   $5 = \frac{a}{e}$
   $5e = a$

   ANSWER: $a = 5e$ and $b = 1/2$.

   (b) (10 points) Suppose that $f(x) = \frac{1}{2}x^4 + 2x^3 - 9x^2 + 7x - 4$. On what interval(s) is f concave up?

   Concave up means $f''(x) > 0$, so we start by finding $f''(x)$.

   $f'(x) = 2x^3 + 6x^2 - 18x + 7$
   $f''(x) = 6x^2 + 12x - 18$

To find where $f''(x) > 0$, we start by finding where $f''(x) = 0$.

$$6x^2 + 12x - 18 = 0$$
$$6(x^2 + 2x - 3) = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3, 1$$

Now we check around these points to see where $f''(x) > 0$:

$f''(-4)$ is positive.
$f''(0)$ is negative.
$f''(2)$ is positive.

So $f(x)$ is concave up on $(-\infty, -3)$ and $(1, \infty)$.

(c) (10 points) Find $\frac{dy}{dx}$ where $xy = y^2 + 1$.

$$\frac{d}{dx}[xy] = \frac{d}{dx}[y^2 + 1]$$
$$1 \cdot y + x \cdot \frac{dy}{dx} = 2y\frac{dy}{dx}$$
$$y = 2y\frac{dy}{dx} - x\frac{dy}{dx}$$
$$y = (2y - x)\frac{dy}{dx}$$
$$\frac{y}{2y - x} = \frac{dy}{dx}$$

(d) (5 points) Suppose that $C(q)$ gives the cost in dollars of producing q text books. Explain in non-math terms what $C'(100) = -50$ means. What are the units of C'?

$C'(100) = -50$ has two possible interpretations that I can think of:

When we have produced 100 text books, the total cost is going down by \$50 per additional text book.

When we have produced 100 text books, making one more will lower the overall cost by \$50.

7. (5 points each) Find the requested information in the following integration problems:

(a) $\displaystyle\int_1^4 t^2 \cdot \frac{t^2+1}{t^4} dt$

$$\int_1^4 t^2 \cdot \frac{t^2+1}{t^4} dt = \int_1^4 \frac{t^2+1}{t^2} dt$$
$$= \int_1^4 (t^2 + 1)t^{-2} dt$$
$$= \int_1^4 (1 + t^{-2}) dt$$
$$= \left[t + \frac{t^{-1}}{-1}\right]_1^4$$

$$= \left[t - \frac{1}{t}\right]_1^4$$
$$= \left[4 - \frac{1}{4}\right] - \left[1 - \frac{1}{1}\right]$$
$$= \left[\frac{4}{1} - \frac{1}{4}\right] - \left[1 - 1\right]$$
$$= \left[\frac{16}{4} - \frac{1}{4}\right] - 0$$
$$= \frac{15}{4} = 3.75$$

(b) $\int \sinh(x)\Big(1 + \cosh(x)\Big)^2 dx$

Since we cannot find this integral using the regular rules, we must use substitution. Let $u = 1 + \cosh(x)$. Then $\frac{du}{dx} = \sinh(x)$. Replacing the expressions in x with expressions with u, we get:

$$\int \sinh(x)\Big(1 + \cosh(x))^2 dx = \int \frac{du}{dx} u^2 dx$$
$$=^2 du$$
$$= \frac{u^3}{3} + C$$
$$= \frac{(1 + \cosh(x))^3}{3} + C = \frac{1}{3}(1 + \cosh(x))^3 + C$$

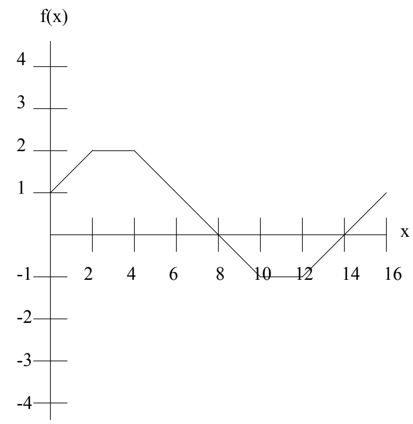
(c) Use the graph of f(x) below to find $\int_4^{16} f(x)dx$.

$\int_4^{16} f(x)dx$ is the area between $f(x)$ and the x-axis starting at $x = 4$ and ending at $x = 16$, counting area above the x-axis as positive and the area below the x-axis as negative.

Area between $x = 4$ and $x = 8$ is $\frac{1}{2}(4)(2) = 4$
Area between $x = 8$ and $x = 10$ is $-\frac{1}{2}(2)(1) = -1$
Area between $x = 10$ and $x = 12$ is $-(2)(1) = -2$
Area between $x = 12$ and $x = 14$ is $-\frac{1}{2}(2)(1) = -1$
Area between $x = 14$ and $x = 16$ is $\frac{1}{2}(2)(1) = 1$

So $\int_4^{16} f(x)dx = 4 + (-1) + (-2) + (-1) + 1 = 1$

Problem 6c



8. (5 points each) Answer the following multiple choice questions. Be sure to justify your answers.

(a) If $f(x)$ is even and continuous for all values of x, then $\int_{-2}^{2} f(x)dx =$
(A) 0 (B) $2\int_0^2 f(x)dx$ (C) cannot say without knowing f.

(B) f is even means that the graph looks the same for negative values of x as for positive values of x. This means that the integral between $x = -2$ and $x = 0$ gives the same answer as the integral from $x = 0$ to $x = 2$, so we can just double that integral.

(b) If $f(x)$ is odd and continuous for all values of x, then $\int_{-2}^{3} f(x)dx =$
(A) $-\int_3^{-2} f(x)dx$ (b) $\int_{-3}^{2} f(x)dx$ (C) cannot say without knowing f.

(A) This is true for all continuous functions.

(c) If f is defined for all values of x, $f'(x) > 0$ for $x < 2$, $f'(2) = 0$, and $f'(x) < 0$ for $x > 2$, then the point $(2, f(2))$ is a
(A) local maximum (B) local minimum (C) inflection point

(A) $f'(x) > 0$ for $x < 2$ means that our graph is increasing before $x = 2$. At $x = 2$, the graph flattens out for a moment, since $f'(2) = 0$. $f'(x) < 0$ for $x > 2$ means that our graph is decreasing after $x = 2$. If we imagine this graph, we see that the point $(2, f(2))$ is a local maximum.

(d) On the graph of g below, $\lim_{x \to 2^-} g(x) =$
(A) -2 (B) 1 (C) 3

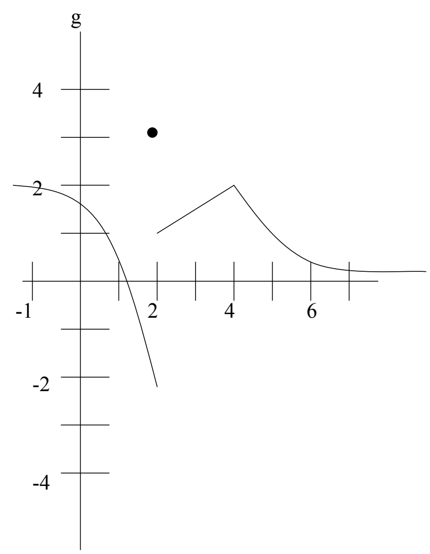
(A) If we start on the left side, the y-value of g is heading toward -2 as x heads toward 2.

(e) On the graph of g below, $\lim_{x \to 4} g(x) =$

(A) 2 (B) 0 (C) DOES NOT EXIST

(A) The y-values of g are heading toward 2 as x heads toward 4 from either side.

Problem 8 d & e



Have a great summer!