UC Merced: MATH 21 — Final Exam — 15 December 2007

On the front of your bluebook print (1) your name, (2) your student ID number, (3) your discussion section number, (4) your room and seat number, and (5) a grading table. Show all work in your bluebook and BOX IN YOUR FINAL ANSWERS where appropriate. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, calculators and crib sheets are not permitted. There are a total of eight problems on both sides of this paper and a total of 150 points. Please start each of the 8 problems on a new page. You have 3 hours to complete the exam.

Some potentially useful information:

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$; $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$; $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

For certain conditions, the following is true: $\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$

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1. (20 points: 2 each) Answer the following Always True (T) or False (F). Only your final answers will be graded on these problems.

   (a) If $\lim_{x\to 5} f(x) = \infty$ and $\lim_{x\to 5} g(x) = -\infty$, then $\lim_{x\to 5}[f(x)+g(x)]$ does not exist.

   (b) $\lim_{x\to 0}\frac{x}{\sin(x)} = \lim_{x\to 0}\frac{1}{\cos(x)} = 1$.

   (c) An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x+2)$.

   (d) If $f'(x) = g'(x)$ for all real numbers x, then $f(x) = g(x)$.

   (e) If $f'(p) = 0$, then the graph $y = f(x)$ must have either a local maximum or a local minimum at $x = p$.

   (f) All continuous functions have derivatives.

   (g) All continuous functions have antiderivatives.

   (h) Of the three hyperbolic functions we studied ($\sinh(x)$, $\cosh(x)$, $\tanh(x)$), $\cosh(x)$ is even and $\sinh(x)$ and $\tanh(x)$ are odd.

   (i) If f is continuous on $[0,7]$ and $f(x) \geq 0$, then $\int_0^7 \sqrt{f(x)}\,dx = \sqrt{\int_0^7 f(x)\,dx}$.

   (j) $\int_{-2}^{1}\frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-2}^{1} = -\frac{3}{2}$.

2. (20 points) Find the requested limits, if they exist. If they do not exist, explain.

   (a) (5 points) $\lim_{x\to 2}\frac{x^2-4}{x-2}$

   (b) (5 points) $\lim_{x\to 2}\frac{4}{x-2}$

   (c) (10 points) $\lim_{x\to\infty} x^2 e^{-x}$

3. (35 points total) Find the requested information on the following unrelated problems.

   (a) (10 points) Using the definition of a derivative, calculate $g'(x)$ where $g(x) = x^2 - x + 2$. **No credit** for using short–cut formulas.

   (b) (5 points) Does the function $f(x) = 2x + |x|$ have a critical point in its domain? Why or why not?

   (c) (5 points) Using the knowledge that $\frac{d}{dx}e^x = e^x$ and that $\ln x$ is the inverse function of e^x, show that $\frac{d}{dx}\ln x = \frac{1}{x}$.

   (d) (7 points) Find the linear approximation (also called the tangent–line approximation) of $f(x) = \arctan(1+x)$ at $x = 0$.

   (e) (8 points) Find $y'(x)$ if $xe^y = y - 1$.

4. (15 points: 5 each) Find the value of the following expressions.

   (a) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left[\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right]\, dx$

   (b) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\, dx$

   (c) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\, dt$

5. (20 points total) Answer the following unrelated integral problems.

   (a) (5 points) Use a Riemann sum with 2 subintervals to approximate $\int_0^4 (x+1)dx$ with the right-endpoint Rule.

   (b) (8 points) Evaluate $\displaystyle\int_1^4 \frac{2}{\sqrt{t}(1+2\sqrt{t})}\, dt$.

   (c) (7 points) Evaluate $\displaystyle\int \frac{xe^x - x + 1}{x}\, dx$.

6. (10 points) Show that of all the rectangles with given area A, the one with the smallest perimeter is a square.

7. (10 points) A baseball diamond is a square with side 90 ft (see below). A batter hits the ball and runs toward first base with a speed of 24 ft/s as shown below. At what rate is his straight-line distance to the **second base** decreasing when he is 45 ft from home plate?

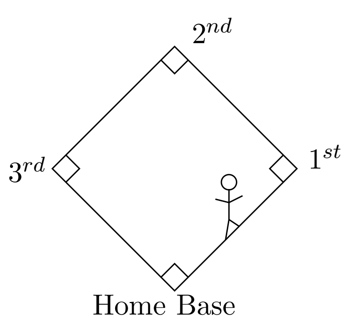
8. (20 points: 5 each) Consider the following graph of $\frac{df}{dx}$, where $f(x)$ is a continuous function with domain $[0, 6]$.

   (a) On what intervals is $f(x)$ increasing? Decreasing?

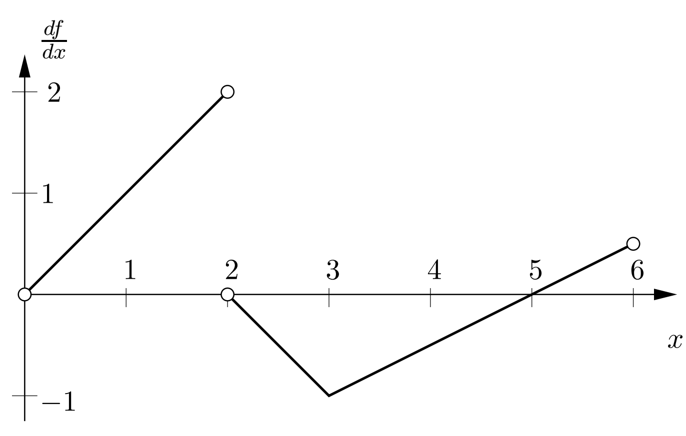
   (b) On what intervals is $f(x)$ concave up? Concave down?

   (c) Find all local maximum and minimum points.

   (d) Carefully sketch a possible graph of $f(x)$ on its entire domain using the information from above.



(a) Problem 7



(b) Problem 8