Math 22 Calculus II

Unit 2 Exam – Solutions

1. (i)

$$\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^x}\,dx$$

$$= \lim_{a\to-\infty} \int_{1+e^a}^{2} \frac{1}{u}\,du \qquad u = 1+e^x$$

$$= \lim_{a\to-\infty} \ln|u| \Big|_{1+e^a}^{2}$$

$$= \lim_{a\to-\infty} \ln 2 - \ln(1+e^a)$$

$$= \ln 2 - \ln 1$$
$$= \ln 2$$

(ii)

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{c\to2^-} \int_0^c \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \lim_{c\to2^-} \sin^{-1}\frac{x}{2}\Big|_0^c$$

$$= \lim_{c\to2^-} \sin^{-1}\frac{c}{2}$$

$$= \sin^{-1} 1$$
$$= \pi/2.$$

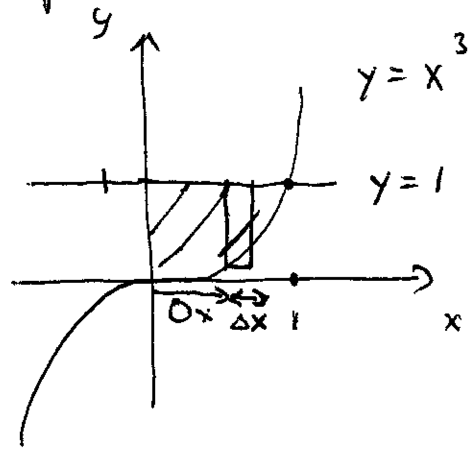
2. Now $-1 \le \cos x \le 1$

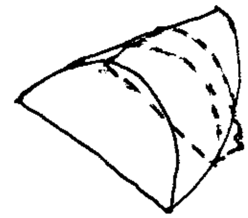
and so

$$0 < \frac{1}{x} \le \frac{2+\cos x}{x} \qquad \text{for } x \ge 1$$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, so does the original integral.

3.

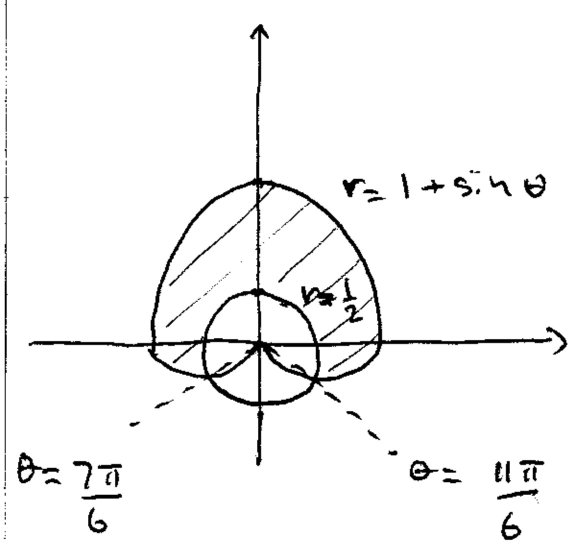




$$\text{Volume of slice} \simeq \frac{1}{2}\pi\left[\frac{1}{2}(1-x^3)\right]^2 \Delta x$$
$$= \frac{1}{8}\pi(1-x^3)^2 \Delta x$$

$$\text{Total volume} = \int_0^1 \frac{1}{8}\pi(1-x^3)^2\,dx$$
$$= \frac{1}{8}\pi \int_0^1 (1-2x^3+x^6)\,dx$$
$$= \frac{1}{8}\pi\left[x - \frac{1}{2}x^4 + \frac{1}{7}x^7\right]_0^1$$
$$= \frac{1}{8}\pi\left(1-\frac{1}{2}+\frac{1}{7}\right)$$
$$= \frac{9\pi}{112}.$$

4.



Intersection points:

$$1 + \sin\theta = \frac{1}{2}$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

$$\text{Area} = 2\int_{\pi/2}^{7\pi/6} \left[\frac{1}{2}\cdot(1+\sin\theta)^2 - \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 \right] d\theta$$

$$= \int_{\pi/2}^{7\pi/6} \left(\frac{5}{4} + 2\sin\theta - \frac{1}{2}\cos 2\theta \right) d\theta$$

$$= \frac{5}{4}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta \Big|_{\pi/2}^{7\pi/6}$$

$$= \frac{5\pi}{6} + \frac{7\sqrt{3}}{8}.$$

5. $r = \theta^2 \qquad 0 \le \theta \le \pi$

$$x = r\cos\theta = \theta^2\cos\theta$$
$$y = r\sin\theta = \theta^2\sin\theta$$

$$\frac{dx}{d\theta} = 2\theta\cos\theta - \theta^2\sin\theta$$
$$\frac{dy}{d\theta} = 2\theta\sin\theta + \theta^2\cos\theta$$

$$\text{Arc length} = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{(2\theta\cos\theta - \theta^2\sin\theta)^2 + (2\theta\sin\theta + \theta^2\cos\theta)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{4\theta^2 + \theta^4}\, d\theta$$

$$= \int_0^{\pi} \theta \sqrt{4+\theta^2}\, d\theta \qquad u = 4 + \theta^2.$$

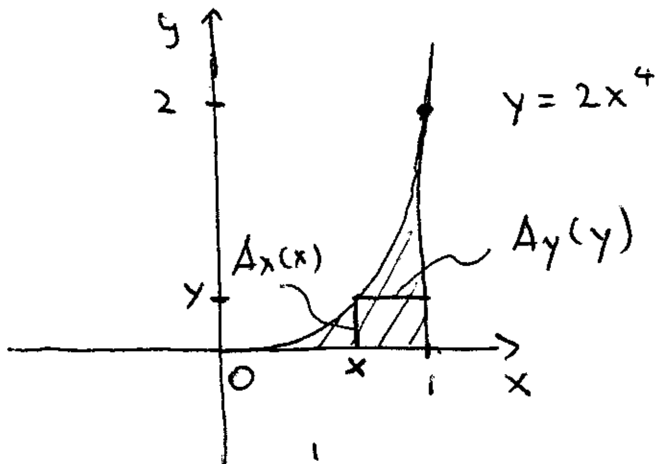
$$= \frac{1}{2} \int_4^{4+\pi^2} \sqrt{u}\, du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_4^{4+\pi^2}$$

$$= \frac{1}{3} (4+\pi^2)^{3/2} - \frac{8}{3}.$$

$$= \frac{1}{3} \left[(4+\pi^2)^{3/2} - 8 \right].$$

6.



$\delta = 3 g/cm^2.$

(i) $$\text{Area} = \int_0^1 2x^4\, dx$$

$$= \frac{2}{5} x^5 \Big|_0^1$$

$$= \frac{2}{5} \text{ cm}^2.$$

$$\text{Mass} = \text{area} \times \text{density}$$

$$= \frac{2}{5} \cdot 3$$

$$= \frac{6}{5} \text{ g}.$$

(ii)

$$\bar{x} = \frac{\int x \delta A_x(x)\,dx}{\text{Mass}}$$

$$= \frac{\int_0^1 x \cdot 3 \cdot 2x^4\,dx}{\frac{6}{5}}$$

$$= 5\int_0^1 x^5\,dx$$

$$= 5 \cdot \frac{1}{6} x^6 \Big|_0^1$$

$$= \frac{5}{6} \text{ cm}$$

$$\bar{y} = \frac{\int y \delta A_y(y)\,dy}{\text{Mass}}$$

$$= \frac{\int_0^2 y \cdot 3 \cdot \left(1 - \left(\frac{y}{2}\right)^{\frac{1}{4}}\right) dy}{\frac{6}{5}}$$

$$= \frac{5}{2^{5/4}} \int_0^2 \left(2^{1/4} y - y^{5/4}\right) dy$$

$$= \frac{5}{2^{5/4}} \left[\frac{1}{2^{3/4}} y^2 - \frac{4}{9} y^{9/4} \right]_0^2$$

$$= \frac{5}{9} \text{ cm.}$$