SOLUTIONS FOR MATH 22 EXAM 2

For Problems #1-#2, evaluate the integral.

1.

$$\int_1^e t \ln t dt$$

Solution: Use integration by parts with $u = \ln t$ and $dv = tdt$, so $du = (1/t)dt$ and $v = t^2/2\, dt$. Using this substitution, we get

$$\int_1^e t \ln t dt = \int_1^e \frac{t}{2} dt = \left.\frac{t^2}{4}\right|_1^e = \frac{e^2-1}{4}$$

.

2.

$$\int_1^{\frac{3}{2}} \frac{dx}{\sqrt{2x-x^2}}$$

Solution: The expression $2x - x^2$ can be written as $-(x^2 - 2x)$. We complete the square on $x^2 - 2x$, getting $(x-1)^2 - 1$. Hence the integral can be written as

$$\int_1^{\frac{3}{2}} \frac{dx}{\sqrt{1-(x-1)^2}}$$

.

Now use the substitution $u = x - 1$, $du = dx$, and change the limits accordingly, to get

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-u^2}}$$

.

This expression is simply $\sin^{-1} u\Big|_0^{\frac{1}{2}} = \frac{\pi}{6}$.

3. Find the partial fraction decomposition of the following function. You do not need to integrate.

$$\frac{x-3}{x^2-4x+4}$$

.

Solution: The denominator has be factored as $x^2 - 4x + 4 = (x-2)^2$. Being a multiple factor, the appropriate form for the partial fraction decomposition is

$$\frac{x-3}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

.

Multiplying both sides of the above equation by $(x-2)^2$, we obtain

$$x - 3 = A(x-2) + B$$

.

This is true for any x, so we can take $x = 2$ to get $-1 = B$. Then we may take $x = 3$ (for instance; any choice will do) to get $0 = A + B = A + -1$. Thus $A = 1$. It follows then that the solution is

$$\frac{x-3}{(x-2)^2} = \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

.

4. Does the following integral converge or diverge. If it converges, find its value.

$$\int_e^\infty \frac{1}{x(\ln x)^3} dx$$

.

Solution: We write the improper integral as a limit:

$$\int_e^\infty \frac{1}{x(\ln x)^3} dx = \lim_{t \to +\infty} \int_e^t \frac{1}{x(\ln x)^3} dx$$

.

If we use the substitution $u = \ln x$, $du = dx/x$, the proper integral becomes

$$\int_1^{\ln t} \frac{1}{u^3} du$$

.

By FTC, we then obtain

$$-\frac{1}{2u^2}\Bigg|_1^{\ln t} = \frac{1}{2} - \frac{1}{2(\ln t)^2}$$

.

Taking the limit as $t \to +\infty$, we get $\frac{1}{2}$. Thus the integral converges to $\frac{1}{2}$.