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Math 22 Spring 2008

Math 22 Midterm 1 Solutions

1. (25 points)

$$\int_0^{\pi/4} x^2 \cos 2x \, dx$$

Solution:

This is just normal integration by parts with the formula given by $\int u\,dv = uv - \int v\,du$. We shall make the following substitutions:

$u = x^2$

$du = 2x\,dx$

$dv = \cos 2x\,dx$

$v = \frac{1}{2}\sin 2x$

Thus, $\Rightarrow \frac{1}{2}x^2 \sin 2x - \int x \sin 2x\,dx$

Now, we use integration by parts a second time with the following substitutions:

$u = x$

$du = dx$

$dv = \sin 2x\,dx$

$v = -\frac{1}{2}\cos 2x$

Thus, $\Rightarrow \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x + \int -\frac{1}{2}\cos 2x\,dx$

$\Rightarrow \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{2}\int \cos 2x\,dx$

$$\boxed{\Rightarrow \frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x}$$

$$\boxed{\Rightarrow \left[\frac{1}{2}x^2 \sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x\right]_0^{\pi/4}}$$

Note:
$\sin(\pi/2) = 1$
$\sin(0) = 0$
$\cos(\pi/2) = 0$
$\cos(0) = 1$

Plugging in zero yields 0

Thus,

$\frac{1}{2}\frac{\pi^2}{16}\sin(\pi/2) + \frac{1}{2}\frac{\pi}{4}\cos(\pi/2) - \frac{1}{4}\sin(\pi/2)$

$\Rightarrow \boxed{\frac{\pi^2}{32} - \frac{1}{4}}$ or $\boxed{\frac{1}{4}\left(\frac{\pi^2}{8} - 1\right)}$

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2. (25 points)

$$\int 4x\cos^4(x^2)\,dx$$

This is quite the tricky integral, but by means of substitution, this integral will become simpler to solve. To start, you can make a primary substitution to simplify the integral.

For this case, I will let $u = x^2$ meaning that $du = 2x\,dx$ (Note: you can use any variable for substitution as long as the appropriate substitution has been made).
Thus, after substitution, the resulting integral becomes:

$$\Rightarrow \int 2\cos^4(u)\,du$$

Recall that $\cos^2 x = \frac{1}{2}(1+\cos 2x)$; This property will help us simplify the integral even more.
If $\cos^2 u = \frac{1}{2}(1+\cos 2u)$, then $\cos^4 u = \frac{1}{4} + \frac{1}{2}\cos 2u + \frac{1}{4}\cos^2 2u$; This is another property we can use to help us.

Thus $\Rightarrow \int 2\cos^4(u)\,du = \int\left(\frac{1}{2}\cos^2 2u + \cos 2u + \frac{1}{2}\right)du$

We can make one more substitution with the property of $\cos^2 x = \frac{1}{2}(1+\cos 2x)$. With that, $\cos^2 2x = \frac{1}{2}(1+\cos 4x)$ by means of using our double-angle formula.
Thus, $\int\left(\frac{1}{4} + \frac{1}{4}\cos 4u + \cos 2u + \frac{1}{2}\right)du \Rightarrow \int\left(\frac{3}{4} + \frac{1}{4}\cos 4u + \cos 2u\right)du$
Now this is just a simple trigonometric expression in which we know how to integrate.

$$\Rightarrow \int\left(\frac{3}{4} + \frac{1}{4}\cos 4u + \cos 2u\right)du = \frac{3}{4}u + \frac{1}{16}\cos 4u + \frac{1}{2}\cos 2u$$

We are not done yet. We started with *x* and ended *u*. Therefore we need to back-substitute with *x*.

$$\therefore \boxed{\frac{3}{4}x^2 + \frac{1}{16}\cos 4x^2 + \frac{1}{2}\cos 2x^2 + C}$$

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3. (25 points)

$$\int_0^{\pi/4} \sin(6x)\sin(7x)\,dx$$

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Solution:
Simplest method to compute this integral is by the following property:

$\boxed{\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]}$ with our $A = 6$ and $B = 7$

Thus, $\int_0^{\pi/4} \sin(6x)\sin(7x)\,dx = \frac{1}{2}\int_0^{\pi/4} \cos(-x) - \cos(13x)\,dx$ Note: $\cos(-x) = \cos x$

$$\Rightarrow \boxed{\frac{1}{2}\int_0^{\pi/4} \cos(x) - \cos(13x)\,dx = \frac{1}{2}\left[\sin x - \frac{\sin 13x}{13}\right]_0^{\pi/4}}$$

$\sin(13\pi/4) = -\sqrt{2}/2$
$\sin(\pi/4) = \sqrt{2}/2$
$\sin(0) = 0$

$$\Rightarrow \frac{1}{2}\left(\frac{\sqrt{2}}{2} - \frac{-\frac{\sqrt{2}}{2}}{13}\right) - (0-0) \Rightarrow \frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{26}\right) = \boxed{\frac{7\sqrt{2}}{26}}$$ (After simplification)

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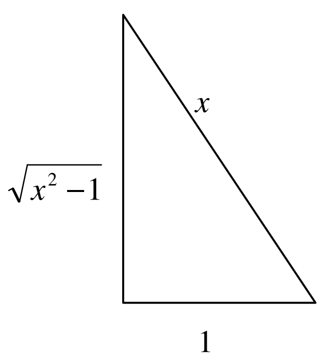
4. (25 points)

$$\int \frac{dx}{(x^2-1)^{5/2}}$$

Solution:
Using the rules for trigonometric substitution, we shall let $\boxed{x = \sec\theta}$ thus, $\boxed{dx = \sec\theta\tan\theta\,d\theta}$. Please see the right triangle below for more details.



Now we substitute everything into the integral to have a new integral written in terms of θ.

$\int \frac{\sec\theta\tan\theta}{(\sec^2\theta - 1)^{5/2}}\,d\theta$ We require the relationship that $\tan^2\theta + 1 = \sec^2\theta$, $\sec^2\theta - 1 = \tan^2\theta$

$$\Rightarrow \int \frac{\sec\theta\tan\theta}{(\sec^2\theta-1)^{5/2}}\,d\theta = \int \frac{\sec\theta\tan\theta}{(\tan^2\theta)^{5/2}}\,d\theta = \int \frac{\sec\theta\tan\theta}{(\tan^5\theta)}\,d\theta = \int \frac{\sec\theta}{\tan^4\theta}\,d\theta$$

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The next best thing to do for this case is rewrite $\tan\theta$ & $\sec\theta$ in terms of $\sin\theta$ & $\cos\theta$

$$\Rightarrow \int \frac{1/\cos\theta}{\sin^4\theta/\cos^4\theta}d\theta = \int \frac{\cos^3\theta}{\sin^4\theta}d\theta = \int \frac{(1-\sin^2\theta)}{\sin^4\theta}\cos\theta\, d\theta \text{ (Used } \sin^2 x + \cos^2 x = 1)$$

Now we can use u- substitution by the following: $u = \sin\theta$, $du = \cos\theta\, d\theta$

Please note the following steps from transforming the integral and back substitution twice:

$$\int \frac{1}{u^4} - \frac{1}{u^2}du \Rightarrow -\frac{1}{3}\frac{1}{u^3} + \frac{1}{u} = -\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta}$$

From the triangle above and our original substitution of $x = \sec\theta$, thus, $\sin\theta = \frac{\sqrt{x^2-1}}{x}$

Therefore, plugging in and adding our constant C, the end result becomes:

$$\boxed{-\frac{x^3}{3\left(\sqrt{x^2-1}\right)^3} + \frac{x}{\sqrt{x^2-1}} + C}$$

(Extra Credit, 5 extra points for correct answer with correct explanation)

$$\int_0^{2\pi} \sin^{25} x\, dx$$

There are several ways to do this. I shall show two types of methods: conceptual and u-substitution.

Conceptual Method:

No matter what power that sine is taken to, on revolution of sine goes above and below the x-axis. There is one shape above the axis and one shape of the exact size below the axis. By taking the areas under the curve, there will be a negative area that cancels out with the positive area, thus $\int_0^{2\pi} \sin^{25} x\, dx = \boxed{0}$

Also, the integrand is odd for the revolution, so the integral will result in 0 anyway.

U-Substitution Method:

I shall rewrite the integral in terms of sine and cosine because the power of sine is odd.

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$$\int_0^{2\pi} \sin^{25} x\,dx \Rightarrow \int_0^{2\pi} (1-\cos^2 x)^{12} \sin x\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$x = 0, u = 1$

$x = 2\pi, u = 1$

$$\Rightarrow -\int_1^1 (1-u^2)^{12}\,du$$

The bounds for the *u*-substitution are the same, thus, the integral results in $\boxed{0}$