Math 22 Midterm 2: Spring 2008 Solutions

1. (15 points) Write the correct form of the partial fraction decomposition of the rational function below. Find the numerator of the terms whose denominator is \((x+1)^2\)

\[
\frac{x^2 + x + 3}{(x+1)^2(x^2 + 5)}
\]

Solution:

1. We need the form of the decomposition, in other words, the setup before finding the constants in any partial fractions situation.

\[
\frac{x^2 + x + 3}{(x+1)^2(x^2 + 5)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 5}
\]

Afterwards, you are asked to find the numerator of the term whose denominator is \((x+1)^2\), in other words, \(B\).

To do so, we need to set a common denominator of \((x+1)^2(x^2+5)\), which will be seen below in order to equate the numerators:

\[
x^2 + x + 3 = A(x+1)(x^2 + 5) + B(x^2 + 5) + (Cx + D)(x+1)^2
\]

Now, we let \(x = -1\), so the terms with \(A, C,\) and \(D\), cancel out (in other words, or go to zero)

\[
\rightarrow 3 = 6B \quad B = \frac{1}{2}
\]

2. (15 points) Determine which of the following expressions is correct if one uses Simpson’s Rule to approximate the integral \(\int_{0}^{3} f(x) \, dx\) with \(n = 4\).

\[
\begin{align*}
(a) & \quad \frac{1}{4} \left[ f\left( \frac{3}{8} \right) + 4f\left( \frac{9}{8} \right) + 2f\left( \frac{15}{8} \right) + 4f\left( \frac{21}{8} \right) \right] \\
(b) & \quad \frac{1}{4} \left[ f(0) + 4f\left( \frac{3}{4} \right) + 2f\left( \frac{3}{2} \right) + 4f\left( \frac{9}{4} \right) + f(3) \right] \\
(c) & \quad \frac{3}{8} \left[ f(0) + 2f\left( \frac{3}{4} \right) + 2f\left( \frac{3}{2} \right) + 2f\left( \frac{9}{4} \right) + f(3) \right] \\
(d) & \quad \frac{3}{4} \left[ f(0) + f\left( \frac{3}{4} \right) + f\left( \frac{3}{2} \right) + f\left( \frac{9}{4} \right) \right]
\end{align*}
\]
Solution:
This multiple choice question can be done either by using process of elimination or knowledge of the formula for Simpson’s Rule.

First, we know our bounds range from 0 to 3. So our \( (b-a) = 3 \)

So our \( \Delta x = \left( \frac{b-a}{n} \right) = \frac{3}{4} \) but since Simpson’s Rule requires our \( \Delta x \) to be divided by 3,

\( \Delta x_{\text{Simpson’s Rule}} = \frac{3/4}{3} = \frac{1}{4} \)

This immediately eliminates choices C and D.

From there, the basic approximation start point starts from \( f(a) = f(0) \) for the Simpson’s Rule, rather than \( f(3/8) \), which eliminates choice A, so with all of the other choices eliminated,

\( \therefore \) Choice B is correct

3. (15 points) Determine whether the following improper integral converges. If it does converge, compute the value of the integral.

\[ \int_{0}^{\infty} x^2 e^{-x} \, dx \]

Solution:
3. We need to use integration by parts for the integral:

\[ \int_{0}^{\infty} x^2 e^{-x} \, dx \]

\[ u = x^2 \]
\[ du = 2x \, dx \]
\[ v = -e^{-x} \]
\[ dv = e^{-x} \, dx \]

\[ \int_{0}^{\infty} x^2 e^{-x} \, dx = \frac{x^2}{e^x} + \int_{0}^{\infty} 2xe^{-x} \, dx \]

Now we use integration by parts again,

\[ u = 2x \]
\[ du = 2 \, dx \]
\[ v = -e^{-x} \]
\[ dv = e^{-x} \, dx \]

\[ \int_{0}^{\infty} 2xe^{-x} \, dx = \left[ x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \right] \]

Applying the first bound of infinity requires the use of L’Hospital’s Rule on the first term twice and second term once, but the whole part yields 0 when infinity is applied.
When 0, is applied, both of the first two terms go to 0, but the last one goes to -2 in the following way:

\[ (0 - 0 - 0) - \left( 0 - 0 - \frac{2}{e^0} \right) = 2 \]

\[ \therefore \text{The integral is convergent with a limit of 2.} \]

4. (15 points) Sketch the finite region bounded by the two curves below. Find the area of this region.

\[ x = y^2 - 1, \; x = 1 - y^4 \]

Solution:

This is what the region looks like along with its viewing rectangle:

Now, we integrate with respect to y to find the area by subtracting the left side of the region from the right as shown in the rectangle.

\[ A = \int_{-1}^{1} [(1 - y^4) - (y^2 - 1)] \, dy \]

\[ \Rightarrow A = \int_{-1}^{1} [(1 - y^4) - y^2 + 1] \, dy \]

\[ \Rightarrow A = \int_{-1}^{1} [(1 - y^4 - y^2 + 1)] \, dy \Rightarrow \left[ 2y - \frac{y^5}{5} - \frac{y^3}{3} \right]_{-1}^{1} \Rightarrow (2 - \frac{1}{5} - \frac{1}{3}) - (-2 + \frac{1}{5} + \frac{1}{3}) \]

\[ \Rightarrow \left( \frac{30}{15} - \frac{3}{15} - \frac{5}{15} \right) + \left( \frac{30}{15} - \frac{3}{15} - \frac{5}{15} \right) = \frac{44}{15} \]
5. (15 points) The integral \( \int_0^{\pi} \pi \sin^2 x \, dx \) can be interpreted as the volume of a solid of revolution. Using a geometric interpretation, find the integral in the list below which yields the same value. Note that it is possible to do this problem without computing any integral.

(a) \( \int_0^1 2\pi y \left( \frac{\pi}{2} - \sin^{-1} y \right) \, dy \)

(b) \( \int_0^\pi 2\pi \sin x \, dx \)

(c) \( \int_0^\pi \sin x \, dx \)

(d) \( \int_0^1 \pi \left[ (\pi - \sin^{-1} y)^2 - (\sin^{-1} y)^2 \right] \, dy \)

Solution:

We are given that \( \int_0^\pi \pi \sin^2 x \, dx \) is an expression of volume of revolution.

However, this is just a simple expression for the disc method of rotating the curve of \( f(x) = \sin x \) (0 \( \leq x \leq \pi \)) about the x-axis.

We can use the method of cylindrical shells to do evaluation the same volume of revolution as we were given.

Choice C seems to just be the method of finding the area under the curve of \( f(x) = \sin x \) (0 \( \leq x \leq \pi \)). We needed a volume expression so that is eliminated.

For the method of cylindrical shells, we use the corresponding y-values of the graph of \( f(x) = \sin x \). Choice B is incorrect because of both the wrong bounds.

Choice A shows that the bounds for rotation for normal xy rotation about the x-axis are 0 to \( \pi/2 \). However, they have doubled the resulting rotation with the correct radius. With shells, rotation about the x-axis uses y bounds.

Therefore, Choice A is correct.

6. (15 points) Suppose the curve \( C \) can be represented in two ways: \( y = f(x), a \leq x \leq b; \) or \( x = g(y), c \leq y \leq d \). Let \( S \) be the surface of revolution obtained by rotating \( C \) around the x-axis. Which two of the integrals below give the surface area of \( S \)?
Solution:
This can also be done by either process of elimination or knowledge of the manipulations of the surface area of revolution formulas. We can eliminate Choice B because the expression $2\pi x\pi$ is used for rotation about the $y$-axis. In addition, Choice A is incorrect because that is also rotation about the $x$-axis. Choice C is correct because of the correct bounds and notation for the specified axis of rotation. With the other choices either chosen or eliminated, Choice D is the remaining correct one because is also in the format of Choice C in simplified Leibniz notation.

∴ Choices C and D

7. (10 points) Write out, but do not evaluate, the integral whose value is the length of the curve $x = e^{-y^2}$, $0 \leq y \leq 1$.

Solution:
For this problem, we can use the formula for arc length in terms of $y$, which is

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Since our bounds for $y$ in terms of $x$ are known (0 and 1), we need to just differentiate $x = e^{-y^2}$ with respect to $y$ by using the Chain Rule and then square our resulting derivative.

Thus, $\frac{dx}{dy} = -2ye^{-y^2}$, so

$$\left(\frac{dx}{dy}\right)^2 = 4y^2e^{-2y^2}$$

From there, we just substitute with our solved information:

$$\Rightarrow L = \int_0^1 \sqrt{1 + 4y^2e^{-2y^2}} \, dy$$