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Math 22 Spring 2008

Math 22 Final Exam Solutions

Problem 1:

For the parametric curve defined below, find the points on the coordinate axes that the curve intercepts, as well as the points on the curve with horizontal or vertical tangent lines. Use this information to sketch the curve.

$$x = t^3 - t$$
$$y = t^2 - 1$$

Solution:

We shall solve for the intercepts first. When we find our *t* points of interest, we shall substitute them back into our given parametric equations in order to find *xy* coordinates.

$x = 0$
$0 = t^3 - t \Rightarrow t(t^2 - 1) = 0$
$\therefore t = 0, \pm 1$

$y = 0$
$0 = t^2 - 1 \Rightarrow (t-1)(t+1) = 0$
$\therefore t = \pm 1$

Now we plug back in:
$$\boxed{\begin{aligned} t = 0 &\Rightarrow (x, y) = (0, -1) \\ t = -1 &\Rightarrow (x, y) = (0, 0) \\ t = 1 &\Rightarrow (x, y) = (0, 0) \end{aligned}}$$

From this we know as *t* is increasing or decreasing, there is some type of loop involved in the motion given by this parametric equation. We shall continue in finding more information.

We shall now find the relationship as *t* goes to both ∞ and $-\infty$ for both *x* and *y*.

$$\boxed{\begin{aligned} \lim_{t\to\infty} x &= +\infty \\ \lim_{t\to-\infty} x &= -\infty \end{aligned}}$$
$$\boxed{\begin{aligned} \lim_{t\to\infty} y &= +\infty \\ \lim_{t\to-\infty} y &= +\infty \end{aligned}}$$

We shall now solve for vertical and horizontal tangent lines using the formula for the derivative of parametric equations.

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$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 1}$$

Although we found the equation for the derivative, just looking at this equation does not help, therefore, we shall use certain components of this in order to find the locations of both vertical and horizontal tangent lines.

To find where there are vertical tangent lines, we let $dx/dt = 0$ and we solve for our t points of interest. Similarly, to find where there are horizontal tangent lines, we let $dy/dt = 0$.

Finding vertical tangent lines:

$$dx/dt = 0 \Rightarrow 3t^2 - 1 = 0 \Rightarrow t = \pm\frac{1}{\sqrt{3}}$$

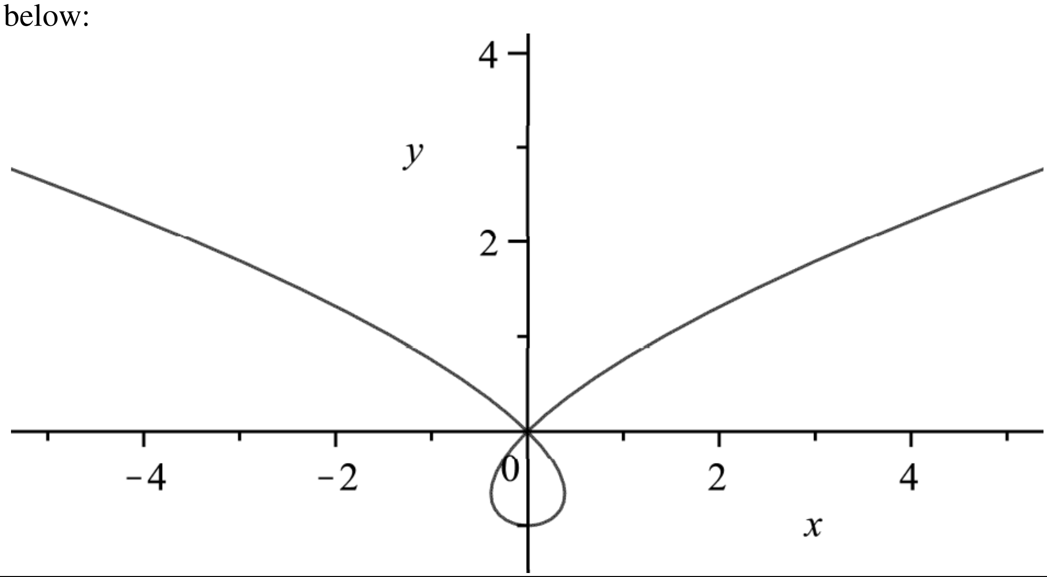
$$t = \frac{1}{\sqrt{3}} \Rightarrow (x, y) = \left(-\frac{2}{3\sqrt{3}}, -\frac{2}{3}\right)$$

$$t = -\frac{1}{\sqrt{3}} \Rightarrow (x, y) = \left(\frac{2}{3\sqrt{3}}, -\frac{2}{3}\right)$$

Finding horizontal tangent lines:

$$dy/dt = 0 \Rightarrow 2t = 0 \Rightarrow t = 0$$
$$t = 0 \Rightarrow (x, y) = (0, -1)$$

We now have enough information in order to draw our picture, which has been provided below:



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Problem 2:

The curve in Problem 1 has a loop. Find the area enclosed by this loop.

Solution:

First off, pay attention to the fact that the drawing must have a loop somewhere. That is probably one of the biggest hints to drawing correctly. Therefore, that means if your drawing does not include something that looks like an evident loop, you might have drawn the picture incorrectly.

Area for a parametric curve is given by the following formula below. Note, when you plug in your information, your resulting integral will be written in terms of the variable *t* (not *x* or *y*).

$$\int_a^b y\,dx = \int_{t_{\min}}^{t_{\max}} y(t)\,(dx/dt)\,dt \qquad \text{Our} \quad \begin{array}{l} y(t) = t^2 - 1 \\ dx/dt = 3t^2 - 1 \end{array}$$

The most difficult component to this problem might be in determining our bounds of *t*. Note that when the curve crosses the origin at *t* = -1, the loop begins and circles back around and crosses the origin at *t* = 1. Therefore the bounds for our loop are -1 and 1.

$$\therefore \int_{t_{\min}}^{t_{\max}} y(t)\,(dx/dt)\,dt \Rightarrow \int_{-1}^{1} (t^2-1)(3t^2-1)\,dt \Rightarrow \int_{-1}^{1} 3t^4 - 4t^2 + 1\,dt \Rightarrow \left[\frac{3}{5}t^5 - \frac{4}{3}t^3 + t\right]_{-1}^{1}$$

$$\Rightarrow \left(\frac{3}{5} - \frac{4}{3} + 1\right) - \left(-\frac{3}{5} + \frac{4}{3} - 1\right) \Rightarrow \left(\frac{6}{5} - \frac{8}{3} + 2\right) \Rightarrow \left(\frac{18}{15} - \frac{40}{15} + \frac{30}{15}\right)$$

$$= \boxed{\frac{8}{15}}$$

Problem 3:

Sketch the polar curve $r = \cos^2\theta$

Solution:

Unless you already know what the curve looks like, the best way to do this is to make a $\theta - r$ list by using angles whose resulting cosine has a radical. By squaring that resulting cosine, the radical is taken off. Please see the chart below for details. Note that you will not have to worry about whether cosine of different angles is positive or negative. Because it is squared, the negative sign eventually disappears. There is a relationship (to be mentioned later) that will save time from plotting points from the chart. From the plotted data points in the chart, we can simply plot points and connect them with a nice smooth curve.

$\theta = 0$	$\theta = \pi/6$	$\theta = \pi/4$	$\theta = \pi/3$	$\theta = \pi/2$
$r = 1$	$r = 3/4$	$r = 1/2$	$r = 1/4$	$r = 0$

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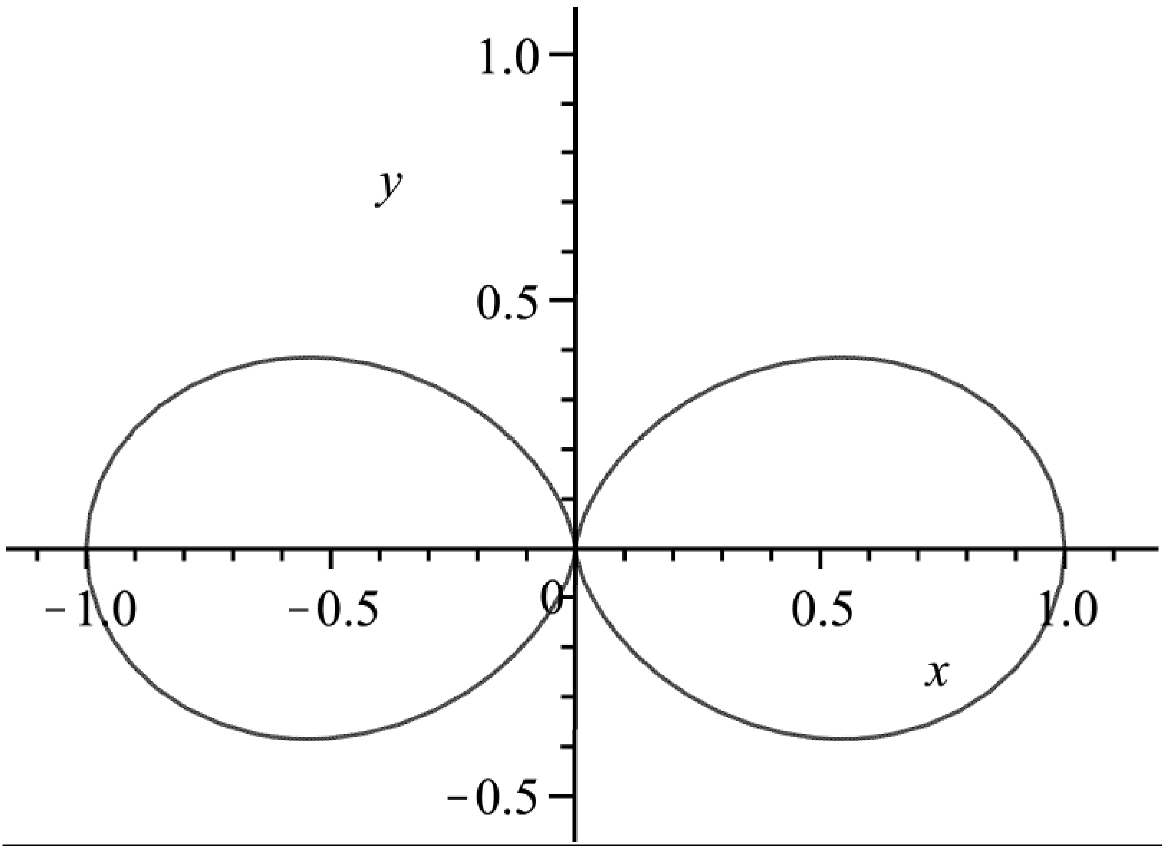
$\theta = 3\pi/2$ $\theta = 3\pi/4$ $\theta = 5\pi/6$ $\theta = \pi$
$r = 1/4$ $r = 1/2$ $r = 3/4$ $r = 1$

Now, you can continue to plug in angles and find their corresponding values, but if you have not noticed already but the equation of the polar curve is even. By plotting half of the curve, the other half is just a mirror reflection (of x –axis) of the top half of the graph.

The picture has been provided below:



Problem 4:

Write out, but do not evaluate, an integral representing the total arc length of the curve in Problem 3.

Solution:

There are numerous ways to do this, but what I shall use is the simpler method dealing with ¼ of the total picture and then quadrupling the integral in order to encompass the entire arc of the picture.

Arc Length for Polar Curves is given by the following:

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$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\theta_{\min}}^{\theta_{\max}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r = \cos^2\theta$$
$$r^2 = \cos^4\theta$$
$$\frac{dr}{d\theta} = -2\cos\theta\sin\theta \quad \text{(Using Chain Rule)}$$
$$\left(\frac{dr}{d\theta}\right)^2 = 4\cos^2\theta\sin^2\theta$$

Since I am using ¼ of the total curve, our bounds range from 0 to $\pi/2$

$\therefore\ 4\int_0^{\pi/2} \sqrt{\cos^4\theta + 4\cos^2\theta\sin^2\theta}\ d\theta$ This is just one of the other answers you may give for full credit. Any manipulation of the bounds requires a different constant of integration on the outside of the integral, but the radicand and $d\theta$ are what must stay the same.

Problem 5:

Find the fourth degree Taylor polynomial of $\cos^2 x$, centered at $x = 0$.

Solution:

Since our $a = 0$, this is a manipulation of the Maclaurin series for $\cos x$.

The Maclaurin series for $\cos x$ is given by the following:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

(Note: when solving 2! = 2, 4! = 24, 6! = 720)

All we must do is take the resulting series as shown as square it as were a polynomial. Since a degree 4 polynomial is needed, we are only taking up to the following series and any powers higher than 4 are disregarded. In a sense, this is exactly like square a polynomial and collecting the like terms up to the 4th degree:

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^2 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{48} + \frac{x^4}{24} - \frac{x^6}{48} + \frac{x^8}{576} + \ldots$$

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$\Rightarrow \boxed{T_4(x) = 1 - x^2 + \frac{x^4}{3}}$ (After collection and reduction of like terms)

Problem 6:

If we approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ by adding together the first N terms, use the alternating series test to find N such that the error is less than one tenth.

Solution:

All we must do is satisfy the following requirement for the series since we know it is convergent:

$|R_n| < b_n$ (Our $b_n = \frac{1}{\sqrt{N+1}}$ for this case)

Thus,

$\frac{1}{10} > \frac{1}{\sqrt{N+1}}$ (And now we solve for N)

$N > 99$ (Since we know this, the next whole number for terms that we have to take is one more than our requirement, since you cannot have a fraction of a term).

$\boxed{\therefore N = 100}$

Problem 7:

Evaluate the following limit.

$$\lim_{x \to 0} \frac{\cos^2 x - 1 + x^2}{x^4}$$

Solution:

If you had done Problem 5 correctly, then this problem would be surprisingly simple. This limit can be done use L'Hospital's Rule, but for the purpose of learning series, this limit requires the application of the Maclaurin series for cos *x*. The other benefit for this problem is that you can use the fourth degree Taylor polynomial for cos *x*. So the limit becomes:

$\lim_{x \to 0} \frac{\cos^2 x - 1 + x^2}{x^4} = \frac{\left(1 - x^2 + \frac{x^4}{3} - ...\right) - 1 + x^2}{x^4}$ (The 1 & x^2 terms cancel out)

$\lim_{x \to 0} \frac{\left(\frac{x^4}{3} - ...\right)}{x^4}$ What you can do here is factor out and x^4 because that is the lowest degree afterwards. (The terms after that have large even powers higher than 4)

This cancels out with the x^4 in the denominator leaving the following:

$\lim_{x\to 0} \frac{1}{3} + \ldots$ Plugging in 0 will make all of the terms after 1/3 go to 0 because they all have a factor of *x* with them leaving only 1/3

Thus, $\lim_{x\to 0} \frac{\cos^2 x - 1 + x^2}{x^4} = \boxed{\frac{1}{3}}$

Problem 8:
Determine whether the following series converges. If it converges find its value.

$$1 - \pi + \frac{\pi^2}{2!} - \frac{\pi^3}{3!} + \ldots$$

Solution:
This is actually and example of a manipulation of the Maclaurin series for e^x.

The Maclaurin series for e^x is given by the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

All we have to do now is find the *x*. We know that it deals with π, but to figure out the *x*, we should notice the pattern. In our given series for e^x, all of the terms are positive; however, since there is an alternation of signs for the following series
$1 - \pi + \frac{\pi^2}{2!} - \frac{\pi^3}{3!} + \ldots$, the *x* is $-\pi$.

Thus, the series is **convergent** with a value of $e^{-\pi}$