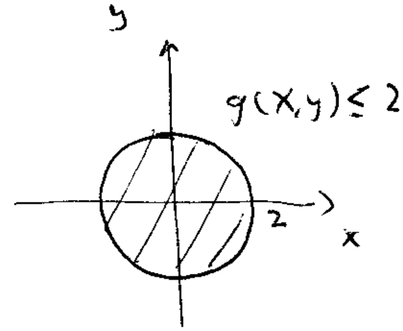
Math 23 Unit 3 Exam Solutions.

1. $$f(x,y) = 4x^2 - xy + 4y^2$$
$$g(x,y) = x^2 + y^2 \le 2$$



Critical points in the interior $g(x,y) < 2$:

$$\vec{\nabla} f = (8x-y)\vec{i} + (8y-x)\vec{j} = \vec{0}$$

i.e.

$$8x - y = 0 \quad \text{and} \quad 8y - x = 0.$$

Solving the system gives $(x,y) = (0,0)$ which lies in the interior.

On the boundary $g(x,y) = 2$:

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$
$$g = 2$$

i.e.

$$8x - y = 2\lambda x \quad \text{— ①}$$
$$8y - x = 2\lambda y \quad \text{— ②}$$
$$x^2 + y^2 = 2 \quad \text{— ③}$$

$y \times$ ① $- x \times$ ② gives $x^2 - y^2 = 0$ or $y = \pm x$.
Substituting into ③ gives $2x^2 = 2$ or $x = \pm 1$.
~~Hence~~ Now the constraint has no endpoints and $\vec{\nabla} g \neq \vec{0}$ on the boundary. Evaluating f at the points we've found we see

$$f(0,0) = 0$$

$$f(-1,-1) = 4\cdot(-1)^2 - (-1)(-1) + 4(-1)^2$$

$$= 7$$

$$f(-1,1) = 4\cdot(-1)^2 - (-1)\cdot 1 + 4\cdot 1^2$$

$$= 9$$

$$f(1,-1) = 4\cdot 1^2 - 1\cdot(-1) + 4\cdot(-1)^2$$

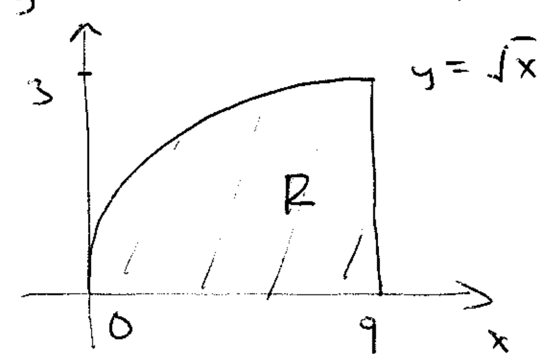
$$= 9$$

$$f(1,1) = 4\cdot 1^2 - 1\cdot 1 + 4\cdot 1^2$$

$$= 7$$

Hence the maximum value of f is 9 and the minimum value is 0.

2. $$\int_0^3 \int_{y^2}^9 y \sin x^2 \, dx\, dy = \iint_R y \sin x^2 \, dA$$



$$\iint_R y \sin x^2 \, dA = \int_0^9 \int_0^{\sqrt{x}} y \sin x^2 \, dy\, dx$$

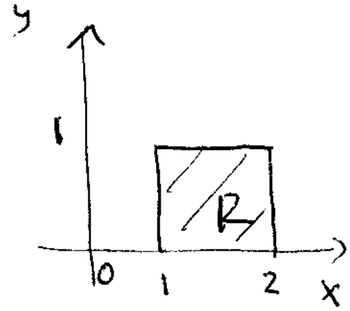
$$= \int_0^9 \left. \tfrac{1}{2} y^2 \sin x^2 \right|_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^9 \tfrac{1}{2} x \sin x^2 \, dx$$

$$= -\frac{1}{4} \cos x^2 \Big|_0^9$$

$$= \frac{1}{4}(1 - \cos 9).$$

3.



$$\text{Volume} = \iiint_W 1 \, dV$$

$$= \iint_R \int_y^{3y} dz \, dA$$

$$= \iint_R z \Big|_{z=y}^{z=3y} dA$$

$$= \iint_R (3y - y) \, dA$$

$$= \iint_R 2y \, dA$$

$$= \int_0^1 \int_1^2 2y \, dx \, dy$$

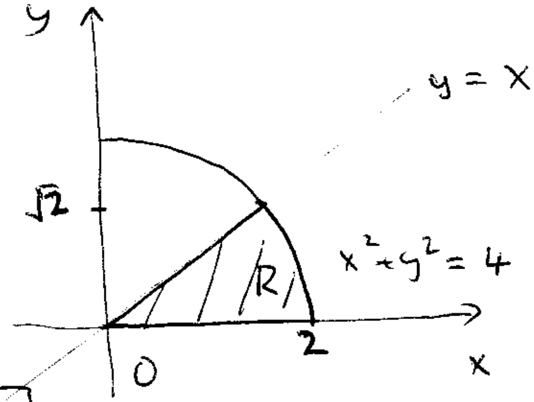
$$= \int_0^1 2xy \Big|_{x=1}^{x=2} dy$$

$$= \int_0^1 (4y - 2y) \, dy$$

$$= \int_0^1 2y \, dy$$

$$= \frac{2}{2} \cdot 1.$$

4.



$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} xy\,dx\,dy = \iint_R xy\,dA$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2} r\cos\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\int_0^{2} r^3 \sin 2\theta\,dr\,d\theta$$

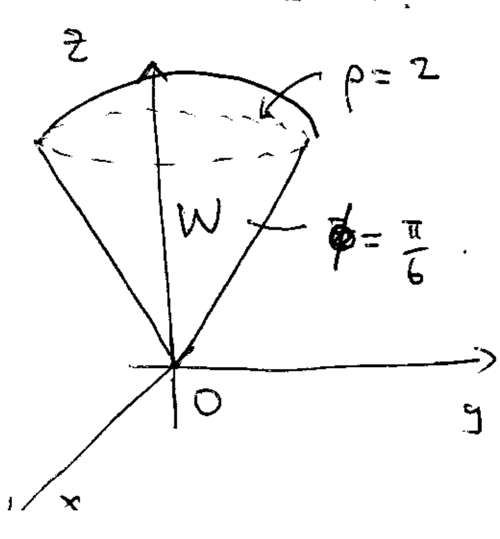
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{4}r^4\sin 2\theta \Big|_{r=0}^{r=2} d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \sin 2\theta\,d\theta$$

$$= 2\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/4}$$

$$= 1.$$

5.



$$\text{Volume} = \iiint_W 1 \, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{6}} \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=2} d\phi$$

$$= \frac{16\pi}{3} \int_0^{\frac{\pi}{6}} \sin\phi \, d\phi$$

$$= \frac{16\pi}{3} \Big[-\cos\phi\Big]_0^{\pi/6}$$

$$= \frac{16\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right).$$

6. (i) We must find a function f such that

$$f_x = y\cos x \quad \text{— (1)}$$
$$f_y = \sin x \quad \text{— (2)}$$

From (1), $f(x,y) = y\sin x + g(y)$. Differentiating w.r.t y and comparing with (2) gives

$$\sin x + g'(y) = \sin x.$$

Hence $g'(y) = 0$ and so $g(y) = C$ and

$$f(x,y) = y\sin x + C.$$

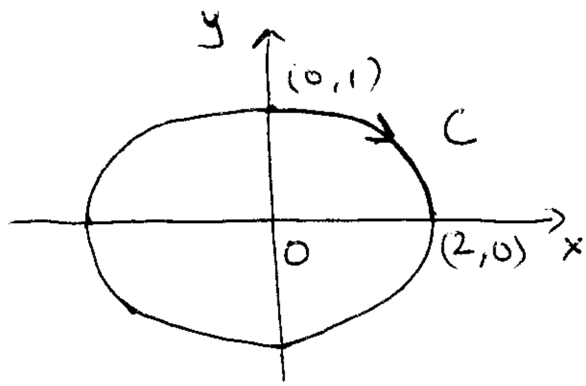
(ii) $$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r}$$

$$= f(1,2) - f(0,0)$$

$$= (2\sin 1 + C) - (0 + C)$$

$$= 2\sin 1 .$$

7.



Parameterize C by $\vec{r}(t) = 2\sin t\,\vec{i} + \cos t\,\vec{j}$ with $0 \le t \le \pi/2$. Then $\vec{r}\,'(t) = 2\cos t\,\vec{i} - \sin t\,\vec{j}$.

Hence,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} (-\cos t\,\vec{i} + 2\sin t\,\vec{j})\cdot(2\cos t\,\vec{i} - \sin t\,\vec{j})\,dt$$

$$= \int_0^{\pi/2} (-2\cos^2 t - 2\sin^2 t)\,dt$$

$$= \int_0^{\pi/2} -2\,dt$$

$$= -\pi .$$