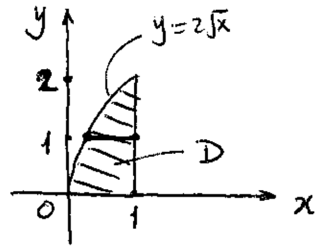
Math 23 Midterm 2 Fall 2007

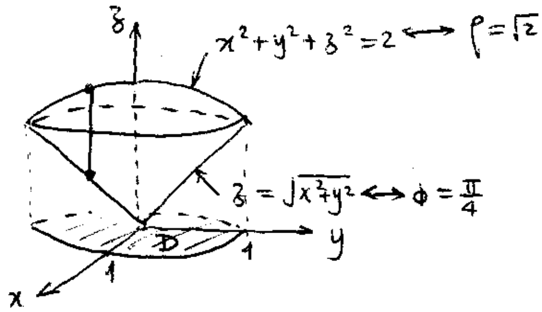
1 (a)



$$y = 2\sqrt{x} \iff \begin{cases} \left(\frac{y}{2}\right)^2 = x \\ y \geq 0 \end{cases} \iff \begin{cases} \frac{y^2}{4} = x \\ y \geq 0 \end{cases}$$

(b) $$M = \boxed{\int_0^2 \int_{y^2/4}^1 \rho(x,y)\, dx\, dy}$$

2.



intersection of $x^2+y^2+z^2=2$ and $z=\sqrt{x^2+y^2}$:

$$x^2+y^2+(x^2+y^2)=2 \Rightarrow \begin{cases} x^2+y^2=1 \\ z=\sqrt{x^2+y^2}=1 \end{cases}$$

(a) $$V = \iint_D \left[\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} 1\, dz\right] dA = \boxed{\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} 1\, dz\, dy\, dx}$$

(b) $$V = \boxed{\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta}$$

(c) $$V = \boxed{\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

3. (a) $$\begin{cases} f_x = y\cos x \\ f_y = \sin x \end{cases} \quad f_x = y\cos x \Rightarrow f(x,y) = y\sin x + h(y) \Rightarrow f_y = \sin x + h'(y)$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = \text{Const.}$$

$$\Rightarrow \boxed{f(x,y) = y\sin(x) + C}$$

(b) $\int_C \vec{F}\cdot d\vec{r} = f(\frac{3\pi}{2}, -1) - f(0,0)$

$$= -1 \sin\frac{3\pi}{2} - 0\sin 0 = \boxed{1}$$

4. (a) $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2y & -(x-2z) \end{vmatrix} = 0\vec{i} - (-1-1)\vec{j} + 0\vec{k}$

$$= \boxed{2\vec{j}}$$

$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(2y) - \frac{\partial}{\partial z}(x-2z) = 0 - 2 - (-2) = \boxed{0}$

(b) $C: \quad x = \cos\theta, \quad y = \sin\theta, \quad z = 0, \qquad 0 \le \theta \le 2\pi$

$d\vec{r} = \langle -\sin\theta, \cos\theta, 0\rangle d\theta$

$\vec{F}|_C = 0\vec{i} - 2\sin\theta\vec{j} - (\cos\theta)\vec{k} = \langle 0, -2\sin\theta, -\cos\theta\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 0 - 2\sin\theta\cos\theta - 0 \, d\theta = -\int_0^{2\pi} 2\sin\theta\cos\theta \, d\theta$

$$= -\int_0^{2\pi} \sin 2\theta \, d\theta = \frac{1}{2}\cos 2\theta \Big|_{\theta=0}^{\theta=2\pi} = \boxed{0}$$

(c) $S_1: \quad x = x, \quad y = y, \quad z = 1 - x^2 - y^2, \quad (x,y)\in D = \{x^2+y^2 \le 1\}$

$$\left.\begin{aligned} \vec{r}_x &= \langle 1, 0, -2x\rangle \\ \vec{r}_y &= \langle 0, 1, -2y\rangle \end{aligned}\right\} \Rightarrow \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$$

$$= 2x\vec{i} + 2y\vec{j} + \vec{k} \quad \text{up!}$$

$\operatorname{curl}\vec{F}|_{S_1} = 2\vec{j}$

$\iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D 4y \, dA = 4\iint_{x^2+y^2\le 1} y \, dA = \boxed{0}$ by symmetry

(d) Yes, because Stokes' Theorem applies

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S = C} \vec{F} \cdot d\vec{r}$$

(e) Let E be the solid bounded by S_1 and S_2. then by the divergence theorem

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV = 0 \,, \quad (1)$$

$\partial E = S_1 - S_2$, so

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} - \iint_{S_2} \vec{F} \cdot d\vec{S} \quad (2)$$

(1) and (2) $\Rightarrow$ $\displaystyle\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_2} \vec{F} \cdot d\vec{S}$.