MATH 24 – EXAM 2 – SOLUTIONS

1. Consider the linear system

$$\begin{aligned} 2x - 10z &= 0 \\ y + 2z &= 1 \\ -4x + 3y + 26z &= 3 \end{aligned}$$

   (a) Write this system in the augmented form $[A|\mathbf{b}]$.

   (b) Find the determinant and determine whether the system has a unique solution.

   (c) Solve this system using Gauss Elimination (RREF).

   (d) What is the span of the solutions (point, line, plane, ...) and what is its dimension?

SOLUTION

(a)

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 0 & -10 & 0 \\ 0 & 1 & 2 & 1 \\ -4 & 3 & 26 & 3 \end{array}\right]$$

(b) $|A| = 0$, so the solution is not unique.

(c)+(d)

$$\left[\begin{array}{ccc|c} 2 & 0 & -10 & 0 \\ 0 & 1 & 2 & 1 \\ -4 & 3 & 26 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1/2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 1 \\ -4 & 3 & 26 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 6 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{RREF}$$

So the solutions are $x - 5z = 0$ and $y + 2z = 1$ or, setting $z = t$, you get $(5t, 1 - 2t, t)$, which is a line, i.e., a 1-dimensional subspace of $\mathbb{R}^3$.

2. Let $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) Calculate AA^T .

(b) Find A^{-1} .

(c) Solve $A\mathbf{x} = \mathbf{b}$ when $\theta = \frac{\pi}{2}$.

(d) Which of the following products are defined:
(i) $A\mathbf{b}$, (ii) $\mathbf{b}A$, (iii) $A^T\mathbf{b}^T$, (iv) $\mathbf{b}^T A$, (v) $\mathbf{b}^T\mathbf{b}$?
Note: you do not need to calculate these products.

SOLUTION

(a) $A^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Using $\cos^2\theta + \sin^2\theta = 1$ gives $AA^T = I$.

(b) Since $AA^T = I$ it follows that $A^{-1} = A^T$.

(c) $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

(d) The ones that are defined are: (i) $A\mathbf{b}$, (iv) $\mathbf{b}^T A$, and (v) $\mathbf{b}^T\mathbf{b}$.

3. Consider the following system of equations in augmented form

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} k & -2 & 1 \\ 2 & -k & -1 \end{array}\right]$$

(a) For which values of k does this system have a unique solution?

(b) Determine the number of solutions for all values of k.

(c) <u>Define</u> the properties of a basis of a vector space. Explain whether the column vectors of A form a basis of $\mathbb{R}^2$ when: (i) $k = 0$ and (ii) $k = 2$.

SOLUTION

(a) $|A| = 4 - k^2$, so the system has a unique solution for all k apart from $k = \pm 2$.

(b) When $k \neq \pm 2$ there is one solution.
When $k = 2$: the system $\left[\begin{array}{cc|c} 2 & -2 & 1 \\ 2 & -2 & -1 \end{array}\right]$ is inconsistent, so it has no solution.
When $k = -2$: the system $\left[\begin{array}{cc|c} -2 & -2 & 1 \\ 2 & 2 & -1 \end{array}\right]$ is consistent, so it has an infinite number of solutions (a line in $\mathbb{R}^2$).

(c) A basis is a set of linearly independent vectors, that spans the vector space.
When $k = 0$ the two columns of A are linearly independent, they span $\mathbb{R}^2$ and are therefore a basis for $\mathbb{R}^2$.
When $k = 2$ the columns of A are linearly dependent, do not span $\mathbb{R}^2$, and are not a basis.

4. Determine whether the following statements are TRUE or FALSE. Note: you must write the entire word TRUE or FALSE. You do not need to show your work for this problem.

(a) The RREF of $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 10 & 0 & 1 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ is the identity matrix.

(b) The vectors $\{[-3,1]\,,\,[2,-3]\,,\,[2,1]\}$ span $\mathbb{R}^2$.

(c) The vectors $\{[-3,1]\,,\,[2,-3]\,,\,[2,1]\}$ form a basis of $\mathbb{R}^2$.

(d) The functions $\{t^2, t^2+1, t^2+3t+1, t+4\}$ are linearly independent.

(e) The set of solutions of the equation $x^2y'' + xy' - y = 0$ forms a vector space.

SOLUTION

(a) TRUE. Since A is upper triangular, $|A| = 2 \cdot 10 \cdot 4 \cdot (-3) \neq 0$, so the RREF is I.

(b) TRUE. For example: it suffices that the first two are independent.

(c) FALSE. 3 vectors in the 2-dimensional $\mathbb{R}^2$ must be dependent, hence they do not constitute a basis.

(d) FALSE. 4 functions in the 3-dimensional $\mathbb{P}_2$ must be dependent. Alternatively, if you calculate the Wronskian, the last (fourth) row is all zero functions, hence the Wronskian is zero and the functions are dependent.

(e) TRUE. It is a linear homogeneous equation, so its solutions are closed under superposition and, therefore, form a vector space.

5. Consider the differential equation $y'' + y' - 6y = 0$.

(a) Find the characteristic equation and roots.

(b) What is the general solution of this differential equation?

(c) Solve the equation when $y(0) = 0$ and $y'(0) = 5$.

(d) What is the long time behavior of the solution?

SOLUTION

(a) Setting $y = e^{rt}$ gives $r^2 + r - 6 = 0$. Hence, $r_1 = 2$ and $r_2 = -3$.

(b) $y_h = c_1 e^{2t} + c_2 e^{-3t}$.

(c) $y(0) = c_1 + c_2 = 0$ and $y'(0) = 2c_1 - 3c_2 = 5$ gives $c_1 = 1$ and $c_2 = -1$. Therefore, $y(t) = e^{2t} - e^{-3t}$.

(d) The solution diverges to infinity.