Math 24 Final Exam: 12–3pm, May 12, 2006 **Instructor: Boaz Ilan**

READ ALL THE INSTRUCTIONS!

1. Write your name on the front of your bluebook as well as in the space below.
   YOUR NAME: ____
2. This exam is closed-book and no calculators are allowed. You are allowed 3 crib sheets (8.×11”). There are EIGHT problems on this exam. You need to solve ALL EIGHT of them (in any order you like). Start each problem on **the top of a new page.**
3. Write a grading grid (enumerated 1..8) on the front of your bluebook.
4. Each problem is worth 20 points for a total of 200 points. To get full credit, you must SHOW YOUR WORK and EXPLAIN ALL THE STEPS IN YOUR SOLUTION. A correct answer with no explanation will not receive credit. The only exceptions are Problems 7 and 8, for which you only need to fill in your answer on your exam sheet.
5. When you are done, hand in your exam sheet and blue book together.

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1. Consider the ordinary differential equation (ODE) $\dfrac{dy}{dt} = y^2 + 2y$.
   (a) Classify this ODE in terms of (1) linear/nonlinear, (2) autonomous/non-autonomous, (3) separable/non-separable? Explain.
   (b) What is an equilibrium solution (in general)? What are the equilibrium solutions for this ODE? Explain your answer.
   (c) What is a stable equilibrium solution (in general)?
   (d) Sketch the phase lines for this ODE in the ty plane and discuss the stability of the equilibrium solutions.

2. Consider the ODE $ty' + y = 1$.
   (a) Solve this equation using the method of Separation of Variables.
   (b) Solve this equation using the method of Integrating Factor.
   (c) What is the equilibrium solution for this equation? Discuss its stability.
   (d) Consider the ODE $tu' = u - u^2$. Make the change of variables $u(t) = \dfrac{1}{y(t)}$ to obtain an ODE for $y(t)$. What is the general solution for $u(t)$?

3. Consider the following three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}.$$

   (a) Are these vectors linearly dependent or independent?

   (b) What is the vector space spanned by these vectors and what is its dimension?

   (c) Find a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ that equals the vector $[4,1,0]^T$.

4. Consider the system of ODEs $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 5 & -2 \\ 8 & -5 \end{bmatrix}$.

   (a) Find the eigenvalues.

   (b) Find two linearly independent eigenvectors.

   (c) Construct two real-valued, linearly independent solutions of this system.

   (d) Using your answer to c) solve the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

5. Consider the ODE $y'' - y = e^{-t}$.

   (a) Find the general solution of the homogeneous equation.

   (b) Find a particular solution of the non-homogeneous equation using the method of Undetermined Coefficients.

   (c) Same as b) using the method of Variation of Parameters.

   (d) Are your answers for b) and c) identical? Explain why.

– OVER –

6. The populations of two species, $x(t)$ and $y(t)$, are described by the following system of ODEs

$$\begin{aligned} x' &= (1-2x)(2-4x-y) \\ y' &= y(6-2x-3y) \end{aligned}$$

   (a) Find the nullclines and sketch them in the xy plane (for $x \geq 0, y \geq 0$).

   (b) Find and list the equilibrium points. Mark them on your sketch.

   (c) Sketch the representative trajectories of the solutions (for clarity, you may want to make a new plot).

   (d) One of the equilibrium points has $x > 0$ and $y > 0$. Linearize the system around this point and determine the stability structure of this point.

   (e) Can the two species co-exist in the long run? Explain.

– OVER –

7. Match the following ODE systems with the phase plane portraits shown in the figures. **Write the matching number in the box provided next to the equation** (no explanation is needed).

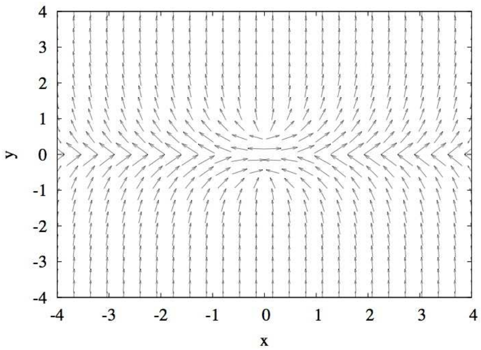
a) $\left.\begin{aligned} x' &= xy \\ y' &= -\cos^2 x \end{aligned}\right\}$ $\square$

b) $\left.\begin{aligned} x' &= \tfrac{x}{y} \\ y' &= x^2 + y^2 \end{aligned}\right\}$ $\square$

c) $\left.\begin{aligned} x' &= -\tfrac{y}{x} \\ y' &= x^2 - y^2 \end{aligned}\right\}$ $\square$

d) $\left.\begin{aligned} x' &= 2 - y^2 \\ y' &= y^2 + xy \end{aligned}\right\}$ $\square$

e) $\left.\begin{aligned} x' &= 2 - y^2 \\ y' &= y + xy \end{aligned}\right\}$ $\square$

1.


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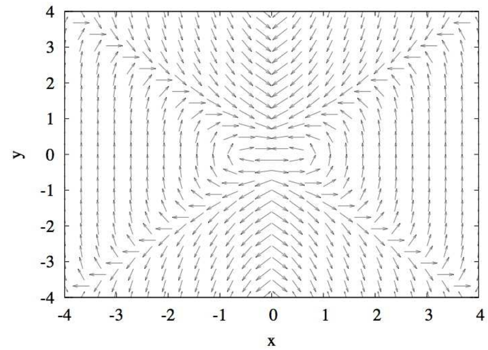

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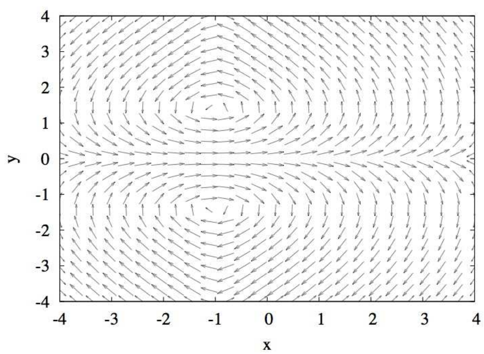

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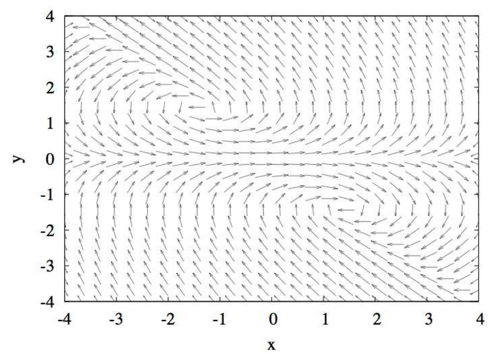

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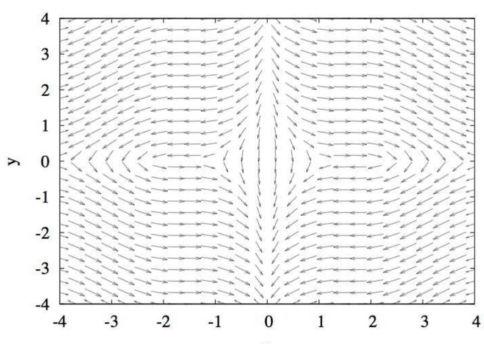

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8. For each question below, **mark either the TRUE or FALSE box.** Each of the 10 questions is worth 2 points. A 5 point bonus will be awarded if you get all of them correctly.

(a) $y(t) = \cos(t)$ is a solution of $y'' + (\sin^2 t + y^2)y = 0$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(b) For the initial value problem $y' = y^2$ with $y(0) = 1$ Picard's Theorem guarantees the existence and uniqueness of a solution. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(c) For the initial value problem $y' = y^2$ with $y(0) = 1$ a solution exists for $-\infty < t < \infty$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(d) If A and B are matrices of size $n \times n$ and A is invertible then $A^{-1}(AB)^T = B^T$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(e) The RREF of $\begin{bmatrix} 1 & 0 & -1 \\ 7 & 2 & 3 \\ 5 & 1 & 0 \end{bmatrix}$ is the identity matrix. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(f) Vectors that form a basis can be linearly dependent. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(g) The ODE $y'' + ty' + t^2 y = 0$ has three linearly independent solutions. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(h) The matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ have the same eigenvalues for any values of a, b, c, d. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(i) The equilibrium solution of the ODE system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} k & 2 \\ -2 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is unstable when $k < 0$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(j) One of the eigenvalues of the ODE $y''' - y'' + y' - y = 0$ is $\lambda = i$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

THE END