MATH 24 – Midterm 1 *Fall Semester 2007*

Duration: 50 minutes
Instructions: Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 50.

1. (8 pts) Consider the differential equation $y'(t) = \frac{y(y-4)}{t+4}$.

   a) If $y(0) = 2$, use Euler's method with time steps $\Delta t = 1$ to approximate $y(1)$ and plot your approximation.

   b) What does Picard's theorem tell you if your initial condition is $y(-1) = 0$?

   c) What does Picard's theorem tell you if your initial condition is $y(-4) = 9$?

   d) Identify all the equilibrium points for $t > 0$ and classify them as stable, unstable or semi-stable.

2. (13 pts) Consider the differential equation $y'(t) - 4ty + 4t = 0$

   a) Classify this equation as precisely as possible.

   b) Find the general solution to this equation.

3. (8 pts) A container is initially filled with 200L of pure milk. To make chocolate milk, some chocolate syrup is then added at the top of the container at a rate of 2L/min. At the same time, pure milk is added at the top of the container at a rate of 3L/min. Meanwhile, the uniformly mixed solution is withdrawn from the bottom of the container at a rate of 6L/min. Write down differential equations and appropriate initial conditions (do not solve) describing:

   a) the time evolution of the volume (V) of chocolate milk (the mixture)

   b) the amount of chocolate syrup (S) in the container.

4. (11 pts) Consider the following matrix and vector

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 8 & -4 & 8 \\ 6 & 2 & 16 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ -4 \\ 8 \end{bmatrix}$$

   a) Find all the solutions to $A\vec{x} = \vec{b}$. If no solutions exists, explain why.

   b) Give a basis for the span of the columns of A (that is, find a basis of col(A)).

5. (10 pts) Answer the following questions in no more than two lines of text (much less is actually needed if you are right on point). Minimal (if any) numerical computation is required.

   a) Consider the system of equations $x' = f(x, y)$, $y' = g(x, y)$. In which direction in the xy-plane are solutions going at a point where $g(x, y) = 0$ and $f(x, y) > 0$?

   b) Describe or draw the geometric interpretation corresponding to a system of 2 equations and 2 unknowns which has no solutions.

   c) Draw or give a mathematical expression which is an example of a one-dimensional vector subspace of $\mathbb{R}^2$.

   d) Simplify the following matrix expression: $BC(AC)^{-1}(BA^T)^T$

   e) Write a 3×3 matrix containing no zeros whose determinant is zero.