Math 24

Final Exam: May 15, 2007

ON THE FRONT OF YOUR BLUEBOOK WRITE (1) YOUR NAME, (2) A **TEN**-PROBLEM GRADING GRID. **Show ALL of your work and EXPLAIN your answers** in your bluebook. A correct answer, but without explanation, will receive no credit. You are allowed three pages of crib sheets. Each problem is worth 10 points for a total of 100 points.

1. Consider the differential equation
$$\frac{dy}{dx} + y = e^{-2x} \ .$$
   (a) Classify this equation as best you can.
   (b) Solve this equation with initial conditions $y(0) = -1$.
   (c) Verify (by direct substitution) that the solution satisfies the initial-value problem.

2. Consider the differential equation
$$\frac{dy}{dt} = (1-y)^3(2-y)^2 \ .$$
   (a) Classify this equation as best you can.
   (b) Define "equilibrium solution" and find all them for this equation.
   (c) Sketch the phase lines for this equation in the (t, y) plane and classify the stability of each equilibrium solution.
   (d) Describe the behavior of the solution with the initial conditions $y(1) = 1.5$.

3. Consider the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.
   (a) Define "linear dependence of vectors", "span of vectors", and "basis of a vector space".
   (b) Determine whether the vectors above are linearly dependent.
   (c) Find the span of these vectors.
   (d) Determine whether the vectors above form a basis for $\mathbb{R}^3$. Explain.
   (e) If possible, find a linear combination of these vectors that equals $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$.

4. Consider the equation $y'' - 9y = e^{3t}$.
   (a) Find the general solution of the homogeneous equation.
   (b) Solve the non-homogeneous equation.

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5. Consider the equation $y'' - 6y' + 9y = \dfrac{e^{3t}}{t}$.

   (a) Find the general solution of the homogeneous equation.

   (b) Solve the non-homogeneous equation.

6. Answer the following TRUE/FALSE questions. You MUST write the entire word TRUE or FALSE. Only your final answer will be graded on this problem.

   (a) The equation $y' - 2ty = 2t$ is separable.

   (b) All the equilibrium solutions of $y' = -\sin(y)$ are stable.

   (c) Picard's theorem guarantees the local existence and uniqueness of the solution of $y' = t\ln y$ with $y(1) = 0$.

   (d) Three vectors in $\mathbb{R}^2$ can be linearly independent.

   (e) One of the eigenvalues of $y''' - 2y'' + 2y' - y = 0$ is -1.

7. Consider the system of equations $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$.

   (a) Find the eigenvalues and eigenvectors.

   (b) Find the general solution of this system.

   (c) Determine the stability structure of the equilibrium solution.

8. Consider the system of equations $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$.

   (a) Find the eigenvalues and eigenvectors.

   (b) Find the general solution of this system.

   (c) Determine the stability structure of the equilibrium solution.

9. The rabbits and bobcats of Merced meet at the city hall and sign a pat to regulate their populations according to the system of differential equations

$$\begin{aligned} \frac{dR}{dt} &= R - RB\ , \\ \frac{dB}{dt} &= -B + kRB\ , \end{aligned}$$

   where $R(t)$ and $B(t)$ are the population numbers of rabbits and bobcats, respectively, and k is a constant to be decided upon.

   (a) Find and sketch the nullclines and equilibria solutions of this system for $k = 1$.

   (b) <u>Linearize</u> the system around each equilibrium solution for $k = 1$ and use this to determine the stability structure of each solution.

   (c) What will happen in the long run if $k = 1$ and the initial populations are $R = B = 10$?

   (d) For which values of k (if any) can the two species coexist?

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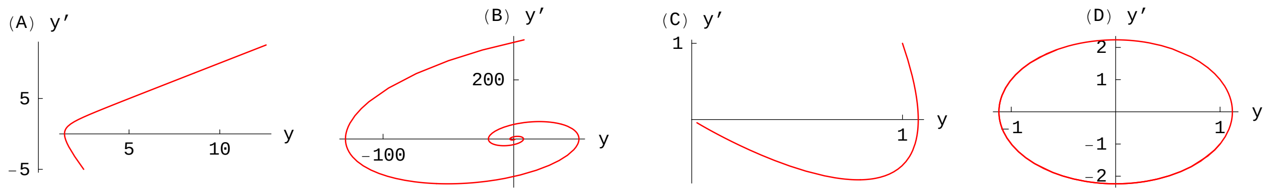
10. Match each of the four systems differential equations with the appropriate phase-plane trajectory graph. Only your final answer will be graded on this problem.

   (a) $y'' - 2y' = 1 + t$
   (b) $y'' - 2y' + y = e^t$
   (c) $y'' - 2y' + y = te^t$
   (d) $y'' + 4y = \cos 2t$



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