EXAM 1, MATH 24, FALL 2008

Instructions: Do not begin the exam until you are instructed to do so. You may write on the exam sheet, but ONLY what is written in your bluebook will be graded.

For each problem, you must show all work in order to receive credit. Partial credit will be given when appropriate, even if the final answer is not correct, but an answer with no work shown will receive zero credit regardless of correctness. You may not use any text, notes, or calculators on this exam, and collaboration is not allowed. Values can be expressed in symbolic form when appropriate, e.g. 2π, e^2, $\sin 1$, etc. Decimal representations are not required. However, trigonometric functions of basic angles, e.g. $\tan\frac{\pi}{4}$, $\sin\pi$, etc. should be evaluated.

1. (10 points each) Classify the following differential equations by determining (i) if they are linear/nonlinear, (ii) their order, (iii) if they are homogeneous/inhomogeneous (iv) if they have constant/variable coefficients. Parts (ii), (iii), and (iv) should only be answered if the equations are linear. Find the general solution to the DE **if** it is linear and first-order.

a. $y' + 2ty = 2t$
Linear, first-order, inhomogeneous, variable coefficient.

$$\begin{aligned}
\mu &= e^{\int 2t\,dt} = e^{t^2} \\
&\Longrightarrow y'e^{t^2} + 2te^{t^2}y = 2te^{t^2} \\
&\Longrightarrow \frac{d}{dt}(ye^{t^2}) = 2te^{t^2} \\
&\Longrightarrow ye^{t^2} = \int 2te^{t^2}dt = \int e^u\,du = e^{t^2} + C \\
&\Longrightarrow \boxed{y = 1 + Ce^{-t^2}}
\end{aligned}$$

b. $y'' + t^2 = 0$
Linear, second-order, inhomogeneous, constant coefficient.

c. $yy' = 8te^t$
Nonlinear due to yy' term.

d. $y' - \frac{y}{t} = 0$
Linear, first-order, homogeneous, variable coefficient.

$$\begin{aligned}
y' &= \frac{y}{t} \\
&\Longrightarrow \int \frac{dy}{y} = \int \frac{dt}{t} \\
&\Longrightarrow \ln|y| = \ln|t| + C_1 \\
&\Longrightarrow y = e^{\ln|t| + C_1} = Ce^{\ln|t|} \\
&\Longrightarrow \boxed{y = Ct}
\end{aligned}$$

2. (5 points) For the direction field of a first-order differential equation $y' = f(t,y)$, explain why ALL horizontal lines are isoclines if $f(t,y)$ is independent of t, that is, if $f(t,y) = f(y)$.

All horizontal lines have a unique y-value. Thus, since $f = f(y)$, the function f will have a unique value at all points on any horizontal line. Since $f = y'$, the slope at each point on any horizontal line will be the same.

3. (30 points) The half-life of the 235-isotope of Uranium (^{235}U) is approximately 7×10^8 years. *Your answers must be numeric (i.e. no variables), but you do NOT have to carry out the arithmetic for either part of this problem.*

(a) What is the decay constant (k) for ^{235}U ?

The model for this problem is the decay IVP $\begin{cases} Q' = kQ, \ k < 0 \\ Q(0) = Q_0 \end{cases}$, *which has solution*

$$Q(t) = Q_0 e^{kt}.$$

Thus, we can rewrite the solution as

$$\frac{Q}{Q_0} = e^{kt}.$$

This is true for all values of t. When $t = t_h$ (the half-life), $Q = \frac{1}{2}Q_0$, since there is half of the original amount remaining at that time. Thus,

$$\frac{\frac{1}{2}Q_0}{Q_0} = \frac{1}{2} = e^{kt_h}.$$

Solving this for k, we get

$$k = \frac{\ln \frac{1}{2}}{t_h} = \frac{\ln \frac{1}{2}}{7 \times 10^8 \text{ years}}$$

FYI, this is equal to roughly -9.9×10^{-10} per year.

(b) How long will it take for a sample of ^{235}U to decay to one-tenth of its original amount?

Now that we have determined k, we can plug this expression into the solution to find the time required for this decay. Let's say that at time T, the sample has decayed to $\frac{1}{10}$ of the original amount. Then, when $t = T$, $Q = \frac{1}{10}Q_0$. Thus, we have

$$\frac{\frac{1}{10}Q_0}{Q_0} = \frac{1}{10} = e^{kT}.$$

Solving this for T, we obtain

$$T = \frac{\ln \frac{1}{10}}{k} = 7 \times 10^8 \cdot \frac{\ln \frac{1}{10}}{\ln \frac{1}{2}} \text{ years}$$

PLEASE NOTE THAT THE LOGARITHMIC QUOTIENT CANNOT *BE SIMPLIFIED!!!*

FYI, this is roughly equal to 2.3 billion years!! This is one of the many reasons that disposal of radioactive waste products is so difficult.

4. (25 points) Solve the initial value problem $\begin{cases} y' + \frac{2y}{t} = t+1 \\ y(2) = 2 \end{cases}$.

Using integration factor method,

$$\mu = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

Thus,

$$\begin{aligned} t^2y' + 2ty &= t^3 + t^2 \\ \frac{d}{dt}(t^2y) &= t^3 + t^2 \\ t^2y &= \frac{1}{4}t^4 + \frac{1}{3}t^3 + C \end{aligned}$$

and thus the general solution is

$$y(t) = \frac{1}{4}t^2 + \frac{1}{3}t + \frac{C}{t^2}$$

Using the initial condition to determine C, we have

$$\begin{aligned} y(2) &= 1 + \frac{2}{3} + \frac{1}{4}C = 2 \\ C &= \frac{4}{3} \end{aligned}$$

Thus, the unique solution to the IVP is

$$\boxed{y(t) = \frac{1}{4}t^2 + \frac{1}{3}t + \frac{4}{3t^2}}$$

Alternatively, we can use variation of parameters. To do so, we solve the corresponding homogeneous equation to obtain y_h:

$$y_h' + \frac{2}{t}y_h = 0$$

This equation is separable, and we obtain

$$y_h = \frac{C}{t^2}$$

Now, replacing constant C with varying parameter $v(t)$, we must solve

$$y_p = \frac{v(t)}{t^2}$$

to obtain a particular solution. Plugging this into the DE (and remembering to use the product rule!), we obtain

$$\begin{aligned} \frac{1}{t^2}v' - \frac{2}{t^3}v + \frac{2}{t^3}v &= t+1 \\ \frac{1}{t^2}v' &= t+1 \\ v' &= t^3 + t^2 \\ v &= \frac{1}{4}t^4 + \frac{1}{3}t^3 \end{aligned}$$

Recall that no constant of integration is required for the varying parameter because it is already included in the homogeneous solution.

$$y_p = \frac{v}{t^2} = \frac{1}{4}t^2 + \frac{1}{3}t$$

We've now obtained a homogeneous and particular solution. Recall that the inhomogeneous principle states that the full solution to the DE is the sum of the two. Thus,

$$\begin{aligned} y &= y_p + y_h \\ y(t) &= \frac{1}{4}t^2 + \frac{1}{3}t + \frac{C}{t^2}, \end{aligned}$$

which is the same function obtained by the integration factor method. Applying the initial condition yields the same solution to the IVP as before.