EXAM 2, MATH 24, FALL 2008

Instructions: Do not begin the exam until you are instructed to do so. You may write on the exam sheet, but ONLY what is written in your bluebook will be graded.

For each problem, you must show all work in order to receive credit. Partial credit will be given when appropriate, even if the final answer is not correct, but an answer with no work shown will receive zero credit regardless of correctness. You may not use any text, notes, or calculators on this exam, and collaboration is not allowed.

1. (20 pts) Solve the following linear system. You may use any method that involves matrices.

$$\begin{aligned} x + y + z &= 1 \\ 2x \quad\;\; + z &= 5 \\ x - y - z &= 0 \end{aligned}$$

$$\boxed{x = \tfrac{1}{2},\ y = -\tfrac{7}{2},\ z = 4}$$

2. For each of the four items below, please write a clear and concise written response.

a. (5 pts) Explain why an equilibrium point in a phase plane must always coincide with intersection of a v-nullcline and an h-nullcline.

A point in equilibrium must have zero spatial derivative in all directions. The intersections of v- and h-nullclines represent points in which both of the spatial derivatives are zero, and are thus equilibria.

b. (5 pts) Explain why the solution space of a non-homogeneous linear system can not possibly be a vector space.

The solution space of a non-homogeneous linear system does not contain the zero vector (since the zero vector does not solve a non-homogeneous system), and thus cannot be a vector space.

c. (5 pts) Explain what a *singular matrix* is, in terms of its determinant and inverse matrix.

A singular matrix has a determinant of zero, and its inverse matrix does not exist.

d. (5 pts) Find the *rank* and *nullity* of the matrix $\begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and identify which matrix elements are pivots.

The rank (number of pivot columns in RREF system) is 3, and the nullity (number of columns without pivots in the RREF system) is 2. Elements a_{11}, a_{23}, a_{34} (in bold) are pivots.

3. (10 pts) Find the inverse of $A = \begin{bmatrix} t & 0 \\ 1 & t^2 \end{bmatrix}$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{1}{t} & 0 \\ -\frac{1}{t^3} & \frac{1}{t^2} \end{bmatrix}}$$

4. (20 pts) Compute $\det(A)$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$

$\boxed{det(A) = -2}$

5. (10 pts) Find all numbers k such that the vectors $\begin{bmatrix} 1 \\ 0 \\ k^2 \\ k \end{bmatrix}$ and $\begin{bmatrix} 4 \\ \tan^{-1}\left[\ln k^6\right] \\ 1 \\ -3 \end{bmatrix}$ are orthogonal.

Scalar product is $4 + k^2 - 3k$. *Vectors orthogonal when scalar product is zero, which occurs when* $\boxed{k = \frac{3 \pm i\sqrt{7}}{2}}$ *by using the quadratic formula.*

6. (20 pts) Find the solution space of

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 5x_1 \qquad - 3x_3 &= 0 \end{aligned}$$

An equivalent system in RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{5} & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 *is arbitrary, so*

$$\begin{aligned} x_3 &= s \\ x_2 &= -\frac{4}{5}s \\ x_1 &= \frac{3}{5}s \end{aligned}$$

and the solution space is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix}, \ s \ \epsilon \ \mathbb{R}}$$

Also, the fractions could be elimated by multiplying the basis vector by 5, and thus

$$\boxed{s \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}}$$

is also a solution space.