EXAM 3, MATH 24, FALL 2008

Instructions: Do not begin the exam until you are instructed to do so. You may write on the exam sheet, but ONLY what is written in your bluebook will be graded.

For each problem, you must show all work in order to receive credit. Partial credit will be given when appropriate, even if the final answer is not correct, but an answer with no work shown will receive zero credit regardless of correctness. You may not use any text, notes, or calculators on this exam, and collaboration is not allowed.

1. For the linear system

$$\begin{aligned} 4x - 2y &= 0 \\ 6x + 3y &= 0 \\ 2x - y &= 0 \end{aligned}$$

a. **(5 points).** Write the system in matrix form.

$$\boxed{\begin{bmatrix} 4 & -2 \\ 6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

b. **(10 points).** Find a basis for the solution space.

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow x = y = 0$$

and thus a basis for the solution space is

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}}$$

i.e. the trivial solution is the unique solution to the system (system is over-determined).

2. A 1-kg mass is attached to a spring with restoring constant 4 N/m and placed on a horizontal surface. The spring is compressed 1 m, then (at $t = 0$) given a "push" (which compresses it further) such that it is given an initial velocity of 2 m/s. Neglect all forms of resistance for parts (a)−(c).

a. **(5 points).** Write down the initial value problem that models the position of the mass.

Using the general model $mx'' + bx' + kx = f(t)$, and the information given in the problem ($m = 1$, $b = 0$ (no resistance), $k = 4$), we have

$$\begin{aligned} x'' + 4x &= 0 \\ x(0) &= 1 \\ x'(0) &= 2 \end{aligned}$$

If you consider compression of the spring to be negative, that's fine, but both IC's must have the same sign.

b. **(5 points).** What is the circular frequency of the mass-spring system, including proper units?

$$\omega = \sqrt{\frac{k}{m}} = 2\frac{rad}{s}$$

c. **(10 points).** Find the position of the mass as a function of time for $t > 0$, i.e. solve the IVP from part (a).

Characteristic equation is $r^2 + 4 = 0$ leading to roots of $\pm 2i$, so $\alpha = 0$ and $\beta = 2$. Using the solution for complex roots

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

we have

$$x(t) = c_1 \cos 2t + c_2 \sin 2t$$

Plugging in $t = 0$ shows that $x(0) = c_1 = 1$. Differentiating and plugging in zero (DON'T FORGET THE CHAIN RULE) shows that $x'(0) = 2c_2 = 2$, so $c_2 = 1$. Thus, the unique solution to the IVP is

$$\boxed{x(t) = \cos 2t + \sin 2t}$$

The additive inverse (i.e. negative) of this solution is also acceptable if your initial conditions were negative.

d. **(10 points).** If resistance is introduced to the system such that the damping constant is 5 kg/s, what would be the general solution for the position as a function of time? You do not need to consider the initial conditions.

The new DE is

$$x'' + 5x' + 4x = 0$$

which has characteristic equation

$$r^2 + 5r + 4 = 0$$

which has roots -4 and -1. Thus,

$$\boxed{x(t) = c_1 e^{-4t} + c_2 e^{-t}}$$

e. **(5 points).** What is the long-term behavior of the system (i.e. as $t \to \infty$) in part (d)?

Both terms in the general solution are decaying exponentials which tend to zero as $t \to \infty$. The spring returns to its equilibrium position in the long term.

3. (10 points). Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$.

Eigenvalues are found by $\begin{vmatrix} 1-\lambda & 4 \\ 0 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$, which has roots 1 and 2. The eigenvalues can also be found by inspection (entries on the main diagonal), since the matrix is upper-triangular.

For $\lambda = 1$, we have

$$\left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

so the second component of the eigenvector is zero, and the first component is arbitrary. Setting the arbitrary component to 1 (by choice, since any non-zero constant is fine), eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda = 2$, we have

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so the second component is arbitrary, and the first component is four times that arbitrary constant. Setting the arbitrary component to 1, eigenvector $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Using the formula

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

we have

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

4. (20 points). Find the general solution to the following DE.

$$y'' + 4y' + 4y = 9e^t$$

The homogeneous equation has characteristic equation $r^2+4r+4=0$, which has repeated root -2. Thus,

$$y_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

The simplest way to solve this problem is by the method of undetermined coefficients, but variation of parameters works as well.

Using undetermined coefficients:

Since the forcing term ($9e^t$) does not solve the homogeneous equation, we try the particular solution $y_p = Ae^t$. The first and second derivatives are also Ae^t, so we have

$$(A + 4A + 4A)e^t = 9e^t$$

and thus $9A = 9$, and $A = 1$. Therefore, the particular solution is $y_p = e^t$ and the general solution (the sum of the homogeneous and particular solutions) is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + e^t$$

5. Explain the validity of the following statements with a short answer, example/counterexample, etc.

a. **(5 points).** Every simple (i.e. distinct, non-repeated) eigenvalue of a non-singular matrix corresponds to a unique eigenvector.

This is false. For a linear system $A\mathbf{x} = \mathbf{b}$, there are NO unique eigenvectors for $\mathbf{b} \neq \mathbf{0}$. Recall that eigenvectors are chosen such that the determinant of $A - \lambda I$ is zero, so that there MUST be free parameters in the system

$$(A - \lambda I)\,\mathbf{x} = \mathbf{0}$$

and thus any eigenvector multiplied by a non-zero constant will also be an eigenvector, meaning that eigenvectors are not unique. Recall in problem 3 that we CHOSE the arbitrary constants to be 1, but any other non-zero choice would have led us to a valid eigenvector as well.

b. **(5 points).** If a matrix transforms a vector such that the resulting vector remains in the same vector space as the original, then the original vector is an eigenvector of the transformation matrix.

This is false. If a matrix transforms a vector such that the resulting vector is PARALLEL (or anti-parallel) to the original, then the vector is an eigenvector of the transformation matrix. Being in the same vector space is not sufficient.

c. **(5 points).** An $n \times n$ linear system $A\mathbf{x} = \mathbf{0}$ is always consistent, and therefore a solution always exists. If the solution is unique, it is only the trivial solution $\mathbf{x} = \mathbf{0}$.

This is completely true. If the determinant of A is zero (i.e. A is singular), then Gauss-Jordan reduction will result in at least one all-zero row. The only way for a system to be inconsistent is for there to be an all-zero row in the reduced row-echelon form of the matrix, with a non-zero element on the right side (the augmented part of the matrix). Since in the system $A\mathbf{x} = \mathbf{0}$, the matrix is augmented with the zero vector, it is not possible to have a non-zero entry on the right side. Thus, the system is always consistent, and a solution always exists.

Another valid reason is that since $\mathbf{x} = \mathbf{0}$ is a solution, a solution obviously exists, so the system must be consistent.

If A is non-singular, it must be row-equivalent to the identity, and there will be no free parameters. Thus, each element of the solution must be identically zero, and the solution must be unique (also implied by non-singularity). Thus, the trivial solution is unique if A is non-singular.

d. **(5 points).** If a 2nd-order linear system of (first-order) DE's has a repeated eigenvalue, then the eigenvectors cannot possibly span the eigenspace, and you must use a generalized eigenvector (i.e. $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{v}$, where λ is an eigenvalue, and $\mathbf{v}$ is its eigenvector) to obtain 2 linearly independent eigenvectors.

This is false. If you obtain two free parameters in the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, you can get two linearly independent eigenvectors from the repeated eigenvalue. Consider the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The repeated eigenvalue is 1, and thus the system becomes

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and both components are arbitrary, leading to $r \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and thus a basis for the eigenspace of the repeated eigenvalue is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, which has dimension 2. Note that we were not required to use the generalized eigenvector **u.** The generalized eigenvector would have been required only if there had been one free parameter instead of two.

FYI: It was no coincidence that the eigenvectors were the column vectors of the original matrix. For any diagonal matrix, the eigenvalues are the entries on the main diagonal, and their corresponding column vectors are their eigenvectors. This is just part of the beauty of diagonal matrices!