MATH 24 – Final Exam

Spring Semester 2008

Instructions. There are a total of 120 points on this final exam. To earn all 120 points, you must provide clear and concise solutions to the problems in this exam in your bluebook. Where applicable, explain your reasoning using complete sentences with proper grammar. Poor presentation may result in loss of credit.

1. For the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix},$$

   (a) (5 points) Find the eigenvalues and eigenvectors.

   (b) (5 points) Verify that the trace of A is equal to the sum of the eigenvalues.

   (c) (5 points) Verify that the determinant of A is equal to the product of the eigenvalues.

2. (10 points) What does Picard's theorem tell us about the following initial-value problem?

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sin y}{t}, \quad y(1) = 1.$$

   What if we prescribed the initial condition $y(0) = 1$ instead?

3. Provide a clear and concise expanation to the questions below.

   (a) (5 points) Suppose that λ is an eigenvalue of A with eigenvector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Show that $\mathbf{v}$ is an eigenvector of $B = A - 7I$ and find the corresponding eigenvalue.

   (b) (5 points) Suppose you have three vectors: $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$. Is it possible that these three vectors do *not* form a basis of $\mathbb{R}^3$ if $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$?

   (c) (5 points) Suppose S is an invertible matrix. What is the determinant of the matrix $S^{-1}AS$?

   (d) (5 points) For what differential equation is the 4th iteration of Euler's method given by the following?

$$y_4 = y_3 + \Delta t \sin(t_3 y_3^4)$$

   (e) (5 points) Suppose the 3×3 matrix A has eigenvalues 1, 2 and 7. What is $\det(A^{-1})$?

4. For the autonomous system of differential equations

$$x' = x - y^2,$$
$$y' = x^2 - y,$$

(a) (5 points) Find the vertical and horizontal nullclines.

(b) (5 points) Find the equilibrium point(s).

(c) (5 points) Classify the linear stability of the equilibrium point farthest away from the origin on the phase-plane, $(x, y) = (0, 0)$.

5. Solve the initial-value problem

$$\ddot{y} + 4y = 2\cos(\omega t), \quad y(0) = \dot{y}(0) = 0, \quad 0 < \omega = \text{constant},$$

for the following two cases:

(a) (5 points) $\omega \neq 2$,

(b) (5 points) $\omega = 2$,

(c) (5 points) Give a brief explanation of the results in terms of a mechanical oscillator.

6. (10 points) For the matrix

$$A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

what are the conditions on a, b, c, d and e needed for A to be invertible?

7. (10 points) Find the solution of the initial-value problem:

$$y' - 2y = 8te^t, \quad y(0) = 1.$$

8. (10 points) Show that the set of solutions of the homogeneous problem

$$y'' + 3y' - 4y = 0$$

is a subspace of the vector function space C^2 (the vector function space of twice continuously differentiable functions). What do we call the span of this set of solutions?

9. (10 points) Find the entries of the third row of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

such that the characteristic polynomial is given by $p(\lambda) = -\lambda^3 + 4\lambda^2 + 5\lambda + 6$.