Math 24 Spring 2009 - Exam 3 Solutions

Prepared by Vikram Rao

Lecturer: Boaz Ilan

1. (32 points) Consider the equation $y'' - 9y = 9e^{-3t}$
   (a) Find the general solution of the homogeneous equation.
   (b) Find the particular solution of the non-homogeneous equation.
   (c) What is the general solution of the non-homogeneous equation and what is its long time behavior?
   (d) Find the solution of the equation that satisfies the following $y(0) = 0, y'(0) = -3/2$

SOLUTION:

(a) We need to find the characteristic roots. Using $y = e^{rt}$ gives $r^2 - 9 = 0, \therefore r = \pm 3$. Thus, $y_h = c_1 e^{3t} + c_2 e^{-3t}$.

(b) We can use one of two methods to find the particular solution: Undetermined Coefficients (U.C.) or Variation of Parameters (VoP). It is probably easier in this case to use U.C. Since we do not want the guess to be similar to the homogeneous solution we need a factor of *t*. Hence, the guess is $y_p = Ate^{-3t}$. This gives

$y_p = Ate^{-3t} \Rightarrow y_p' = Ae^{-3t} - 3Ate^{-3t} \Rightarrow y_p'' = -6Ae^{-3t} + 9Ate^{-3t}$

Plugging back into the DE we get that $A = -3/2. \therefore y_p = -\frac{3}{2}te^{-3t}$.

(c) General Solution: $y = y_h + y_p = c_1 e^{3t} + c_2 e^{-3t} - \frac{3}{2}te^{-3t}$. Inspecting this solution we conclude that, unless $c_1 = 0$, the solution diverges to infinity. If $c_1 = 0$ the solution converges to zero.

(d) When the initial conditions are imposed we get that $c_1 = c_2 = 0$. This means that the solution is $y = -\frac{3}{2}te^{-3t}$.

2. (32 points)
   (a) Define (in general) the eigenvalues and eigenvectors of a matrix.
   (b) Find (and justify your answer) the eigenvalues of $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
   (c) Find the eigenvectors of A.
   (d) What is the dimension of the subspace spanned by all the eigenvectors of A?

SOLUTION:

(a) The eigenvalues and eigenvectors are defined by $Av = \lambda v$, where λ is (in general) a complex number called an eigenvalue and $v \neq 0$ is a non-zero vector, which is called an eigenvector corresponding to λ.

(b) We can find the eigenvalues using $|A - \lambda I| = 0$. Here,

$$|A - \lambda I| = \begin{vmatrix} (3 - \lambda) & 2 & 1 \\ 0 & (0 - \lambda) & 0 \\ 0 & 0 & (3 - \lambda) \end{vmatrix} = 0 \Rightarrow (3 - \lambda)(0 - \lambda)(3 - \lambda) = 0.$$

Therefore the eigenvalues are $\lambda_1 = 0$ & $\lambda_2 = \lambda_3 = 3$.

(c) For $\lambda_1 = 0$ the augmented form of $(A - \lambda_1 I)v = 0$ is $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$,

which gives the eigenvector (up to scalar multiplication) $v_1 = \begin{bmatrix} 1 \\ -3/2 \\ 0 \end{bmatrix}$.

For $\lambda_2 = \lambda_3 = 3$ the augmented form is $\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, for which there is only one eigenvector (up to scalar multiplication) $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This is a deficient case.

(d) The span of v_1 and v_2 is a two-dimensional subspace of $\Re^3$, i.e., a plane.

3. (35 points) Answer the following TRUE/FALSE questions. You must write the entire word TRUE or FALSE. You do NOT need to justify your answers.

(a) The equation $y'' - ky = \cos\left(\sqrt{2}t\right)$ corresponds to a resonant system if $k = 2$.

(b) $y_p = Ate^{-3t}$ is a suitable guess for the particular solution of $y'' + 6y' + 9y = 2e^{-t}$.

(c) $y_p = A\cos\left(t^2\right) + B\sin\left(t^2\right)$ is a suitable guess for the particular solution of $y'' - y = \cos\left(t^2\right)$.

(d) The system $\{x_1' = x_2, x_2' = 2x_2 - 2x_1\}$ has the same eigenvalues as $y'' - 2y' + 2y = 0$.

(e) Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Then A is invertible and the eigenvalues of A^{-1} are 1/2 and 1/3.

(f) If A is a 2×2 matrix with a complex eigenvalue then A is invertible.

(g) If A is a 2×2 matrix and c is any number then the eigenvalues of cA are c times the eigenvalues of A.

SOLUTION:

(a) Beware of the negative sign in front of the ky term: it means that the solutions are exponential rather than sine-cosine. Hence, there cannot be resonance in this system. Indeed, the condition for resonance is $\omega_0 = \omega_f$.

Here $\omega_f = \sqrt{2}$ but $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{-2}{1}} = i\sqrt{2}$ is complex (not a "real" frequency). Hence, the statement is FALSE.

(b) For the homogeneous equation we have a double root of $r = -3$. Since the non-homogeneous term is similar to the homogeneous solution "twice" (once for each root) we need to an additional factor of t^2 *in the guess, that is,* $y_p = At^2e^{-3t}$. So the statement is FALSE.

(c) The statement is FALSE because Undetermined Coefficients works only when the arguments of the sines and cosines are linear functions of t.

(d) When the system is recast as a matrix, the characteristic polynomial is found by the usual condition for eigenvalues, that is,

$$J(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \to \begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0.$$

We see that the system has the same eigenvalues as the characteristic roots of the DE. Thus, the statement is <u>TRUE</u>.

(e) A is upper triangular so its eigenvalues are along the main diagonal, i.e. they are 2, 2, and 3. Since neither is zero it follows that A is invertible and the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, which are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$. So the statement is <u>TRUE</u>.

(f) Since one of the eigenvalues of A is complex the other eigenvalue must be its complex conjugate. Therefore, neither eigenvalue is zero. It follows that A is invertible and the statement is <u>TRUE</u>.

(g) You could first test this with $c = 2$ and $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \& \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}$. The eigenvalues of the first matrix are 1 and 4, while for the 2nd matrix they are 2 and 8, which is twice the eigenvalues of the first matrix. In general, recall that the eigenvalues and eigenvectors are defined by $Av = \lambda v$. Multiplying this by $c \neq 0$ gives $cAv = c\lambda v$. Therefore, the matrix cA has an eigenvalue $c\lambda$. For the special case $c = 0$ the matrix cA is the zero matrix that has (only) zero eigenvalues. This proves that the statement is <u>TRUE</u> in general.