Math 24 Final Exam: 11:30–2:30 pm, May 15, 2009 **Instructor: Boaz Ilan**

READ ALL THE INSTRUCTIONS!

1. Turn off your cell phones and any other device that may emit disturbing noise. This exam is closed-book and no calculators are allowed.
2. Write your name on the front of your bluebook as well as in the space below.

   YOUR NAME: ____

3. There are SEVEN problems on this exam, each worth 200/7 points. Please write a grading grid (enumerated 1..7 and T for total) on the front of your book.
4. Start each problem on the top of a new page. SHOW YOUR WORK and EXPLAIN ALL THE STEPS IN YOUR SOLUTION. Explain your reasoning using complete sentences where applicable. A correct answer with no explanation will not receive credit. The only exceptions are Problems 6 and 7, in which you only need to fill in your answer on your exam sheet.
5. When you are done, insert your exam sheets in your book and hand them together.

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1. Consider the differential equation (DE) $\dfrac{dy}{dt} = \dfrac{y(2-y)}{1+y}$.
   (a) Classify the equation as best you can.
   (b) Find the equilibrium solutions.
   (c) Using phase lines determine the stability structure of the equilibrium solutions.
   (d) What is the long time behavior corresponding to the initial conditions $y(0) = 1$?
   (e) Sketch the phase lines and the solution curves corresponding to the initial conditions $y(0) = 1$ and $y(0) = 3$.

2. Consider the DE $\dfrac{dy}{dt} - \sqrt{t}\, y = e^{\left(\frac{2}{3}t^{\frac{3}{2}}\right)}$.
   (a) Find the Integrating Factor.
   (b) Find the general solution.

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3. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 3 \\ 0 & 4 & -1 \end{bmatrix}$.

   (a) What is the determinant of A?
   (b) What are the solutions of $A\mathbf{x} = \mathbf{0}$?
   (c) What is the RREF of A?
   (d) What is the product of the eigenvalues of A?
   (e) What is the sum of the eigenvalues of A?

4. Consider the DE $y'' - 2y' + y = \dfrac{e^t}{t}$.

   (a) Find the general solution of the homogeneous problem.
   (b) Find a particular solution of the non-homogeneous problem using the method of Variation of Parameters.
   (c) What is the general solution of the DE?

5. Consider the system of DEs $x_1' = 2x_1 + x_2$, $x_2' = x_1 - 3x_2$.

   (a) Write the system in the form $\mathbf{x}' = A\mathbf{x}$.
   (b) Find the eigenvalues of A.
   (c) Find the eigenvectors of A.
   (d) Classify the stability structure of the equilibrium solution.
   (e) Sketch the phase portrait of the solutions in the (x_1, x_2) plane.

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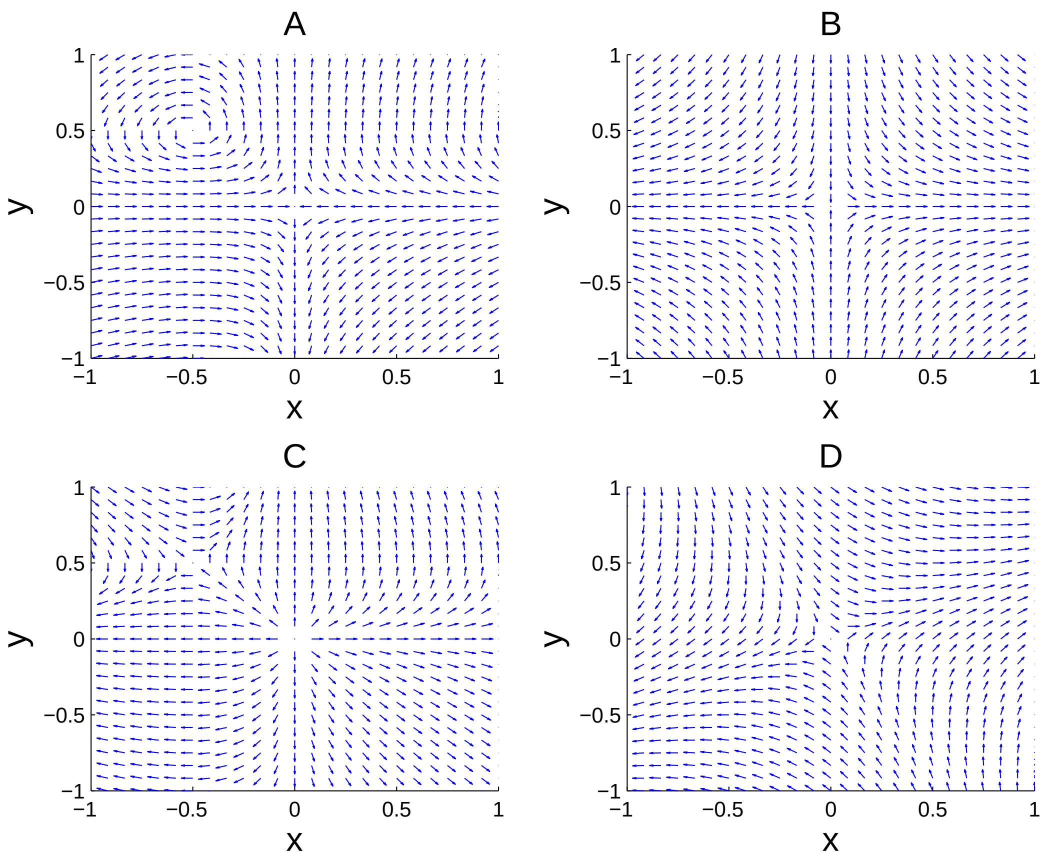
6. Match the following equations (1)–(4) with their corresponding direction fields A–D. You do not need to show your work for this problem. Write your answers on your exam sheet.

(1) $x' = x + y,\ y' = x - y$ corresponds to _____

(2) $x' = x,\ y' = -y$ corresponds to _____

(3) $x' = x(0.5 - y),\ y' = y(0.5 + x)$ corresponds to _____

(4) $x' = -x(0.5 - y),\ y' = y(0.5 + x)$ corresponds to _____



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7. (a) $y(t) = \cos t$ is an equilibrium solution of $y'' + y = 0$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(b) Picard's Theorem guarantees the local existence and uniqueness of a solution to the IVP $y' = y^{3/2},\ y(0) = 0$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(c) The IVP $y' = y^{3/2},\ y(0) = 0$ has a solution that is defined for all time. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(d) If A and B are commuting matrices then $(AB)^2 = BA^2B$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(e) If A and B are $n \times n$ matrices and $\exists (AB)^{-1}$ then $A(AB)^{-1}B = I$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(f) The vectors $\{(1,0,1)^T, (-1,2,1)^T, (0,1,2)^T\}$ span $\mathbb{R}^3$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(g) For three vectors in $\mathbb{R}^3$ to span $\mathbb{R}^3$ they must form a basis for $\mathbb{R}^3$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(h) The functions $1 + t$ and $t(1 + t)$ are linearly dependent. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(i) $y_p = Ae^t$ is a suitable guess for the particular solution of $y'' - 2y' + 2y = e^t$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(j) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ has three linearly independent eigenvectors. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(k) If A is a 3×3 real matrix whose eigenvalues are $\{-1, i, -i\}$ then $\mathbf{x} = \mathbf{0}$ is a stable solution of the system of DEs $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(l) One of the eigenvalues of the DE $y''' - y'' + 2y' - 2y = 0$ is $\sqrt{2}\, i$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(m) The system of DEs $x' = x + y$, $y' = x^4 - y^4$ has a line of equilibrium solutions in the (x, y) plane. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

(n) Let $x' = x + y$, $y' = x^4 - y^4$. The Jacobian matrix of the linearized system around $(x, y) = (1, -1)$ has eigenvalues $\{0, -5\}$. $\boxed{\text{TRUE}}$ $\boxed{\text{FALSE}}$

HAVE A GREAT SUMMER!