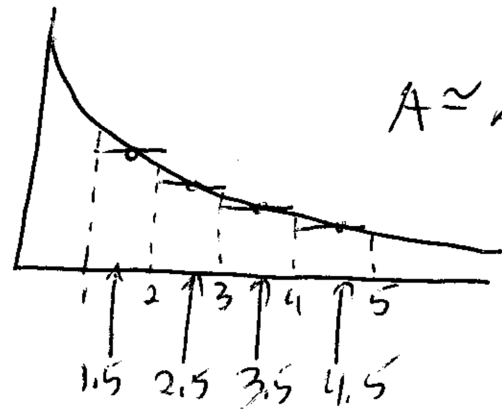
MATH 30 MIDTERM II SOLUTION 11/10/08

① Midpoint approximation of $\int_1^5 \frac{dx}{x+1}$ with $n=4$



$$A \simeq \Delta x \left(\frac{1}{2.5} + \frac{1}{3.5} + \frac{1}{4.5} + \frac{1}{5.5} \right)$$

$$A \simeq \left(\frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \frac{2}{11} \right)$$

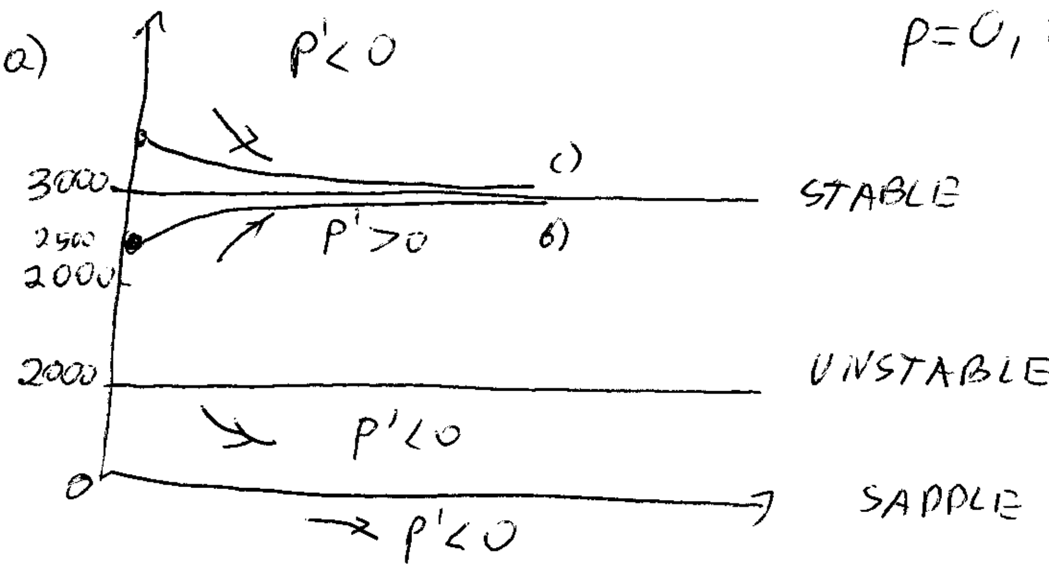
$$\Delta x = \frac{5-1}{4} = 1$$

② $\frac{dP}{dt} = P^2(P-2000)(3000-P)$

Find equilibrium points $\frac{dP}{dt} = 0$

$P = 0, 2000, 3000$

a)



After a long time populations in b) & c) approach 3000 fish

(3) a) $f(x) = \frac{A}{(x+1)^3} \quad x \geq 0 \qquad f(x) = 0 \quad x < 0$

If $f(x)$ is a probability density function

$$\int_{\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^{\infty} \frac{A}{(x+1)^3}\,dx = 1$$

$$\int_0^{\infty} \frac{A}{(x+1)^3}\,dx = \lim_{\ell \to \infty} \int_0^{\ell} \frac{A}{(x+1)^3} = \lim_{\ell \to \infty} \left(\frac{-A}{2} (x+1)^{-2} \Big|_0^{\ell} \right)$$

$$= \lim_{\ell \to \infty} \left(-\frac{A}{2(\ell+1)^2} + \frac{A}{2} \right) = \frac{A}{2}$$

$$\frac{A}{2} = 1 \qquad A = 2$$

b) Find median $\int_m^{\infty} \frac{2}{(x+1)^3} = \frac{1}{2}$

$$\int_m^{\infty} \frac{2}{(x+1)^3}\,dx = \lim_{\ell \to \infty} \int_m^{\ell} \frac{2}{(x+1)^3}\,dx$$

$$= \lim_{\ell \to \infty} \left(-\frac{1}{(x+1)^2} \Big|_m^{\ell} \right) = \lim_{\ell \to \infty} \left(-\frac{1}{(\ell+1)^2} + \frac{1}{(m+1)^2} \right)$$

$$= \frac{1}{(m+1)^2} = \frac{1}{2} \qquad (m+1)^2 = 2$$

$$\boxed{m = \sqrt{2} - 1 \sim .414 \ldots}$$

c) Find average (EXTRA CREDIT)

$$\bar{x} = \int_0^{\infty} x f(x)\,dx = \lim_{\ell \to \infty} \int_0^{\ell} \frac{2x}{(x+1)^3}\,dx$$

use u-sub $u = x+1 \quad du = dx$

$$\bar{x} = 2\lim_{l\to\infty}\left[\int_1^{l+1} \frac{u-1}{u^3}\,du = \int_1^{l+1} u^{-2} - u^{-3}\,du\right]$$

$$\bar{x} = 2\lim_{l\to\infty}\left(-u^{-1} + \frac{u^{-2}}{2}\Big|_1^{l+1}\right)$$

$$\bar{x} = 2\lim_{l\to\infty}\left(-\underbrace{\frac{1}{l+1}}_{\to 0} + \underbrace{\frac{1}{2(l+1)^2}}_{\to 0} + \frac{1}{1} - \frac{1}{2}\right)$$

$$\boxed{\bar{x} = \frac{2\cdot 1}{2} = 1}$$

(4) Find the arc length $y = \frac{2}{3}x^{3/2} \quad 0 \le x \le 3$

$$L = \int_0^3 \sqrt{1+(y')^2}\,dx \qquad y' = x^{1/2}$$
$$(y')^2 = x$$

$$L = \int_0^3 \sqrt{1+x}\,dx \qquad u = 1+x \quad du = dx$$

$$L = \int_1^4 u^{1/2}\,du = \frac{2u^{3/2}}{3}\Big|_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}\text{cm}$$

$L = vt$ (velocity × time)

$$t = \frac{L}{v} = \frac{14}{3}\cdot\frac{1}{.1} = \boxed{\frac{140}{3}\text{ sec}}$$

(5) $2y'' + y' - y = 0 \qquad y = e^{rx}$

$2r^2 e^{rx} + r e^{rx} - e^{rx} = 0$

e^{rx} is never zero so you can divide the whole equation by e^{rx}

$2r^2 + r - 1 = 0$

$(2r - 1)(r + 1) = 0$

$\boxed{r = \frac{1}{2}, -1}$