Applied Math Preliminary Exam: Complex Analysis

University of California, Merced, January 2015

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work.

Problem 1. (5 points each)

(a) Define precisely what it means for a function $f\colon \mathbb{C} \to \mathbb{C}$ to be analytic at z_0.

(b) Prove that $\overline{\sin z} = \sin \overline{z}$.

(c) Using a well-labeled graph, plot the set of all complex numbers satisfying $\left|\dfrac{2}{zi-2}\right| \le 4$.

Problem 2. (5 points each)

(a) Write i^i as $x + iy$, where $x, y \in \mathbb{R}$.

(b) For $z_1, z_2 \in \mathbb{C}$, prove that $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

(c) Write $\left(-\sqrt{3} + i\right)^{11}$ as $x + iy$, where $x, y \in \mathbb{R}$.

Problem 3. (a) (5 points) Find and identify the kind of singularities (pole, removable, or essential) of the function

$$f(z) = \frac{\sin(z^2)}{ze^{1/(z-3)}}.$$

(b) (15 points) Find the Laurent series for

$$f(z) = \frac{1}{(z-4)(z+2)}$$

valid for $2 < |z| < 4$.

Problem 4. (15 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 2} dx.$$

Problem 5. (5 points each)

(a) For the linear fractional transformation

$$f(z) = \frac{az + b}{cz + d},$$

prove that $f(z)$ is not a constant if $ad - bc \neq 0$.

(b) Describe mathematically and draw what happens to the vertical line $x = 2$ under the mapping $f(z) = 1/z$.

(c) Describe mathematically and draw what happens to the half-strip $R = \{z = x + iy\colon 0 \le x \le 1$ and $y \ge 0\}$ under the mapping $f(z) = z^2$.

Problem 6. (5 points each) Short answers.

(a) Give an example of a function that has a singularity at 0 and whose contour integral around the circle $|z| = 1$ is zero. Explain briefly why.

(b) Give an example of functions $p(z)$ and $q(z)$ that are analytic at z_0 and where z_0 is an m-th order zero of $q(z)$ but $f(z) = p(z)/q(z)$ does not have an m-th order pole at z_0. Explain.

(c) Let f be a complex function on real variables, i.e., $f(t) \in \mathbb{C}$ for $t \in \mathbb{R}$. Show that

$$\operatorname{Re}\left[\int_a^b f(t)\ dt\right] = \int_a^b \operatorname{Re}\left[f(t)\right]\ dt.$$

(d) Explicitly state the Cauchy-Riemann equations and explain their significance.