Duration: 240 minutes

Answer all questions. Partial credit will be awarded to correct, but partial work. Points will be deducted for non-sensical answers. This test is meant to be difficult, so you are not expected to be able to answer every question perfectly.

1. Consider the initial-value problem $y' + P(x)y = x$, $y(0) = 1$, where

$$P(x) = \begin{cases} 1 & 0 \le x \le 2, \\ 3 & x > 2. \end{cases}$$

   (a) Comment on the existence and uniqueness of solutions for this initial-value problem.
   (b) Find a *reasonable* solution to this problem, *i.e.* one that is continuous for $x \in [0, \infty)$.
   (c) Sketch the graph of this solution.

2. Solve $y' = \exp(x^2)/y^2$ with initial condition $y(0) = 1$.

3. Solve $y'' - y' - 2y = \cos x - \sin 2x$ with initial conditions $y(0) = -7/20$, and $y'(0) = 1/5$.

4. Solve $y''' + y' = 0$ with initial conditions $y(0) = 0$, $y'(0) = 1$, and $y''(0) = 2$.

5. A generalized Ricatti equation takes the form $y' = P(x)y^2 + Q(x)y + R(x)$.

   (a) Suppose $y = u(x)$ is a solution of this equation. Show that $y = u + 1/v$ reduces the generalized Ricatti equation to a linear equation in v.
   (b) Given that $u(x) = x$ is a solution of $y' = x^3(y - x)^2 + y/x$, use your result from part (a) to find all other solutions to this equation.

6. Consider the van der Pol oscillator governed by the equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0.$$

   (a) Determine how the stability of the zero solution depends on the non-negative parameter, ϵ.
   (b) If a pendulum is governed by this equation, apply your stability result from (b) to describe the behavior of this pendulum.

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7. The equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be *exact* if it can be written in the form $[P(x)y']' + [f(x)y]' = 0$, where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. Find the necessary condition for exactness and then give the method of solution.

8. Locate and classify the singular points of the following differential equations.

   (a) $(x-1)y'' + \sqrt{x}y = 0$, $x \geq 0$.

   (b) $y'' + y' \log x + xy = 0$, $x \geq 0$.

   (c) $xy'' + y \sin x = 0$.

   (d) $(x^2 - x)y'' + xy' + 7y = 0$.

9. Find two linearly independent solutions of $xy'' + (1+x)y' + 2y = 0$, valid near $x = 0$. It is sufficient to obtain the first three nonvanishing terms in the infinite series.

10. Find all eigenvalues and eigenfunctions for the Sturm-Liouville problem

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = y(b) = 0, \quad b > 1.$$