AMGS Preliminary Exam in Ordinary Differential Equations, Jan. 15, 2014

Instructions: Credit will be awarded mainly based on the **quality of your work** and the **clarity of your explanations.** In the course of doing any problem, if you encounter any integrals that cannot be evaluated in closed form, leave your answer *in terms of* those integrals. Throughout this exam, dots denote derivatives with respect to time t.

1. (6 pts) Consider
$$\frac{dx}{dt} = a(t)x(t)$$
where both a and x are scalar functions of t. Derive the general solution $x(t)$ subject to the initial condition $x(0) = x_0$.

2. (12 pts) Let
$$A(t) = \begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix}$$
and
$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$
Derive the solution of
$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}(t)$$
subject to arbitrary initial conditions $\mathbf{x}(0) = [x_1(0), x_2(0)]$.

3. (12 pts) Let $A(t)$ be a general, time-dependent matrix. Consider the system
$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}(t).$$
Let $\exp(M)$ denote the matrix exponential of the square matrix M. Is it true that the general solution of the system is
$$\mathbf{x}(t) = \exp\left(\int_0^t A(s)\,ds\right)\mathbf{x}(0)?$$
If you answer *yes*, then provide a proof. If you answer *no*, then provide a counterexample.

4. (12 pts) Consider the nonlinear equation
$$\ddot{x}(t) + x^5 = 0.$$
Find a function $E(x, \dot{x})$ that satisfies the following property: if $x(t)$ solves the nonlinear equation, then
$$\frac{d}{dt}\left[E(x(t), \dot{x}(t))\right] = 0.$$

5. (16 pts) Consider $\dot{x}(t) = x^\alpha$ with $x(0) \geq 0$ and $\alpha > 0$. Show that when $\alpha > 1$, any solution satisfying $x(0) > 0$ must blow up at some finite positive time t. Next, show that when $\alpha < 1$, there exists an initial condition for which the resulting solutions are not unique. Conclude that $\alpha = 1$ is the only value of α for which the equation has a unique solution that exists globally in time.

6. (14 pts) Use power series to find the general solution $y(x)$ of the equation

$$\frac{d^2y}{dx^2} - xy(x) = 0.$$

For what values of x does the power series solution converge?

7. (14 pts) Solve

$$\ddot{x}(t) + x(t) = \begin{cases} \sin t & 0 \leq t \leq \pi \\ 0 & t > \pi \end{cases}$$

subject to the initial conditions $x(0) = \dot{x}(0) = 0$. What happens if we instead allow the $\sin t$ forcing to persist for all $t \geq 0$?

8. (14 pts) Find the general solutions of the following two equations:

(a) The first-order equation

$$\frac{dy}{dt} = t^{20}y + \cos(t^{14}).$$

(b) The second-order equation

$$t^2\frac{d^2x}{dt} + 20t\frac{dx}{dt} + 14x = 0.$$