Direct and inverse scattering of extended objects

Arnold D. Kim

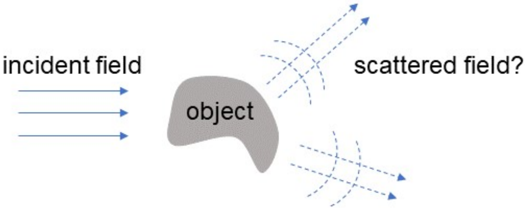
Department of Applied Mathematics, University of California, Merced

Supported by AFOSR and NSF

September 17, 2020

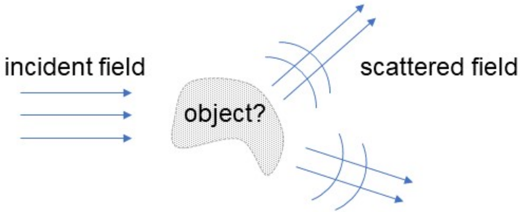
Direct scattering problem

Compute the field scattered by a finite-sized object(s) due to some controlled illumination.



Inverse scattering problem

Reconstruct the finite-sized object(s) from measurements of the scattered field.



Governing PDEs and their fundamental solutions

The "fields" we consider are solutions to the Helmholtz equation,

$$(\Delta + k^2)u = 0,$$

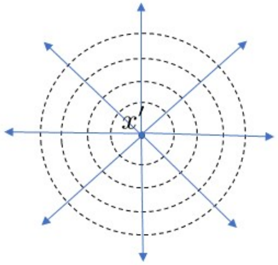
or the negative/modified Helmholtz equation,

$$(\Delta - k^2)u = 0.$$

We make extensive use of the fundamental solution $\Phi(x, x')$ satisfying

$$(\Delta \pm k^2)\Phi = -\delta(x - x'),$$

along with appropriate "outgoing" radiation conditions.



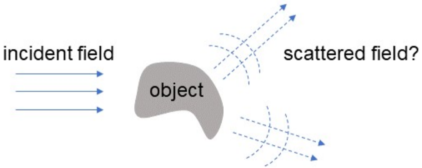
Direct scattering problem for one object

We model one object as a simply connected, open set $D \subset \mathbb{R}^n$ $(n = 2, 3)$ with boundary B. Let $\bar{D} = D \cup B$. The exterior is then $E = \mathbb{R}^n \backslash \bar{D}$.

We consider the following boundary-value problem for (modified) Helmholtz's equation,

$$(\Delta \pm k_0^2)u = 0, \quad \text{in } E,$$
$$(\Delta \pm k_1^2)v = 0, \quad \text{in } D,$$
$$u = v \quad \text{and} \quad \partial_n u = \partial_n v \quad \text{on } B,$$

along with appropriate radiation conditions on u.



Solving the direct scattering problem

We use the Method of Fundamental Solutions (MFS) to solve the direct scattering problem.

- Introduced as a numerical method by Mathon and Johnston (1977).
- Approximate the interior and scattered fields by a superposition of finitely many fundamental solutions, each of which is an *exact* solution of the PDE.
- Strength of each fundamental solution is determined through the boundary conditions.

Exterior solution

The field u exterior to the object satisfies

$$(\Delta \pm k_0^2)u(x) = 0 \quad x \in E.$$

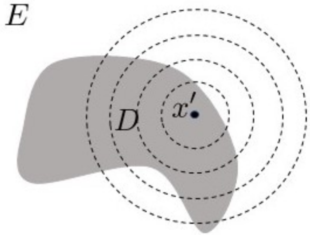
Let $\Phi_0(x, x')$ denote the fundamental solution satisfying

$$(\Delta \pm k_0^2)\Phi_0 = -\delta(x - x').$$

Suppose $x' \in D$. Then Φ_0 satisfies

$$(\Delta \pm k_0^2)\Phi_0 = 0 \quad x \in E.$$

Hence, Φ_0 *exactly satisfies* the PDE governing the exterior solution.



Interior solution

The field v interior to the object satisfies

$$(\varDelta \pm k_1^2)v = 0 \quad \text{in } D.$$

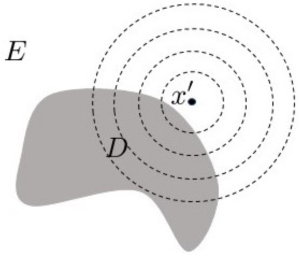
Let $\Phi_1(x, x')$ denote the fundamental solution satisfying

$$(\varDelta \pm k_1^2)\Phi_1 = -\delta(x - x').$$

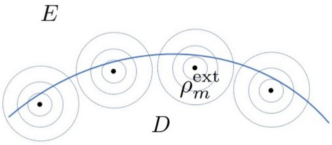
Suppose $x' \in E$. Then Φ_1 satisfies

$$(\varDelta \pm k_1^2)\Phi_1 = 0 \quad x \in D.$$

Hence, Φ_1 *exactly satisfies* the PDE governing the interior solution.



MFS: field approximations

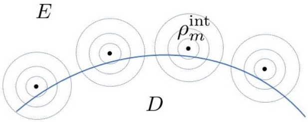


The exterior field with wavenumber k_0 is approximated by

$$u(x) \approx u^{\text{inc}}(x) + \sum_{m=1}^{M} \Phi_0(x, \rho_m^{\text{ext}}) c_m^{\text{ext}}.$$

The interior field with wavenumber k_1 is approximated by

$$v(x) \approx \sum_{m=1}^{M} \Phi_1(x, \rho_m^{\text{int}}) c_m^{\text{int}}.$$



MFS: boundary conditions

Using the approximations,

$$u(x) \approx u^{\text{inc}}(x) + \sum_{m=1}^{M} \Phi_0(x, \rho_m^{\text{ext}}) c_m^{\text{ext}} \quad \text{and} \quad v(x) \approx \sum_{m=1}^{M} \Phi_1(x, \rho_m^{\text{int}}) c_m^{\text{int}},$$

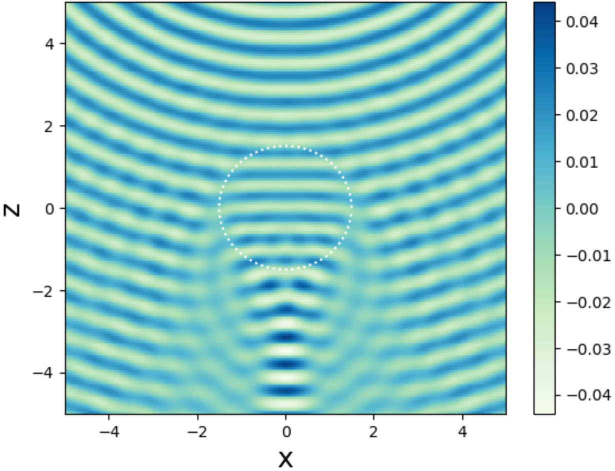
we require

$$u(\rho_m^{\text{bdy}}) = v(\rho_m^{\text{bdy}}) \quad \text{and} \quad \partial_n u(\rho_m^{\text{bdy}}) = \partial_n v(\rho_m^{\text{bdy}}), \quad m = 1, \cdots, M.$$

This collocation method yields a $2M \times 2M$ linear system for the expansion coefficients, c_m^{ext} and c_m^{int} for $m = 1, \cdots, M$.

Example simulation for Helmholtz's equation

Scattering by a dielectric cylinder due to a point source (above).



Comments on MFS for the direct scattering problem

- *Useful for simulating measurements.*

  This method is meshless and allows for easy evaluation wherever measurements are taken.

- *Easy to implement and extend to multiple objects.*

  The key is determining an appropriate set of boundary points and the corresponding unit outward normals.

- *There is an "art" to choosing where to put the exterior/interior points.*

  This choice affects accuracy and conditioning of the linear system.

- *Error analysis is complicated.*

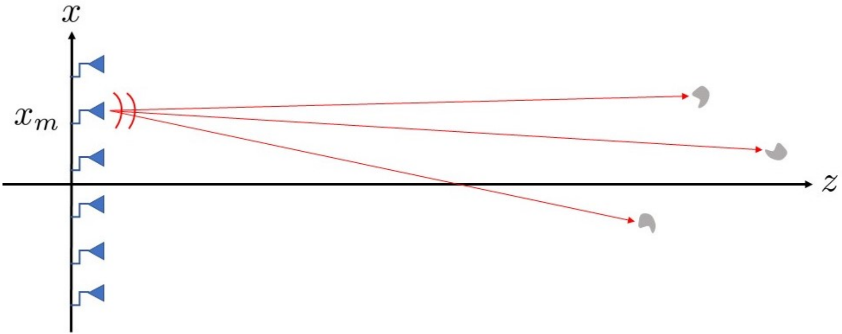
  Empirical results show very high accuracy. Analysis of "convergence" is complicated.

Inverse scattering for array imaging

An array of sources/receivers conduct a suite of experiments leading to the measurement matrix B.

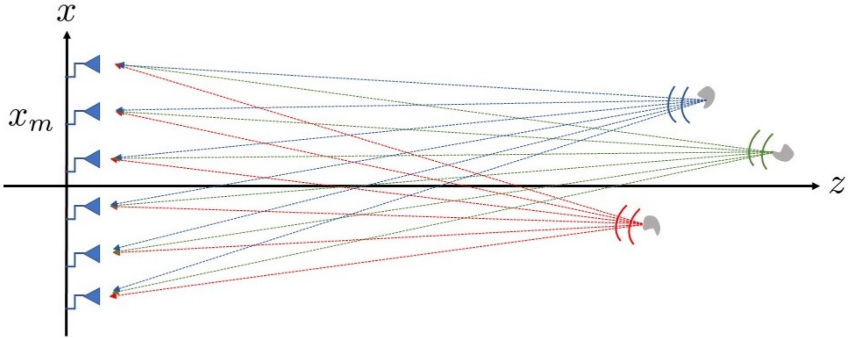
Inverse scattering for array imaging

Each element of the array illuminates the medium.



Inverse scattering for array imaging

All elements of the array measures the scattered fields.



Measurement matrix

- The entire suite of experiments leads to the measurement matrix $B \in \mathbb{C}^{M \times M}$.
- The matrix entry b_{mn} corresponds to a source at array element m measured at array element n.
- The rows of B are manifest from the *spatial diversity at the source* due to illuminations from different spatial locations.
- The columns of B are manifest from the *spatial diversity at the receiver* due to measurements of the scattered field at different spatial locations.

The inverse scattering problem

Given the measurement matrix B, reconstruct properties of the scattering objects in the medium, *e.g.* location, shape, material properties, etc.

- Here we focus on location and shape of objects.
- The inverse scattering problem requires a model for the measurements.
- We introduce a point-based model based on fundamental solutions.
- This point-based model allows us to solve the inverse scattering problem using elementary linear algebra methods.

A little bit of scattering theory

Scattering due to some potential V is governed by

$$(\Delta + k_0^2)u = -k_0^2 V u.$$

Using the fundamental solution, we find that the solution u is given by

$$u = u^{\text{inc}} + k_0^2 \int_D \Phi_0(x, x') V(x') u(x') \mathrm{d}x',$$

which is called the Lippmann-Schwinger equation.

The first Born approximation is given by

$$u \approx u^{\text{inc}} + k_0^2 \int_D \Phi_0(x, x') V(x') u^{\text{inc}}(x') \mathrm{d}x',$$

Born approximation

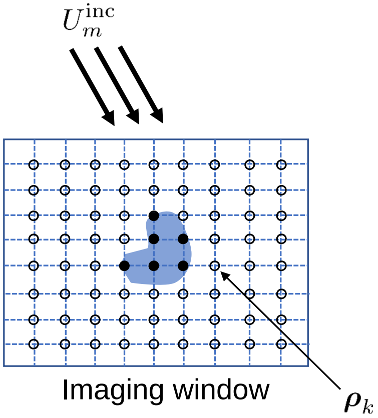
Suppose we apply a numerical quadrature rule to the first Born approximation:

$$
\begin{aligned}
u^{\text{Born}} &= u^{\text{inc}} + k_0^2 \int_D \Phi_0(x, x') V(x') u^{\text{inc}}(x') \,\mathrm{d}x' \\
&\approx u^{\text{inc}} + k_0^2 \sum_{q=1}^{Q} \Phi_0(x, \rho_q) V(\rho_q) u^{\text{inc}}(\rho_q) w_q \\
&\approx u^{\text{inc}} + \sum_{q=1}^{Q} \Phi_0(x, \rho_q) u^{\text{inc}}(\rho_q) \alpha_q .
\end{aligned}
$$

Upon discretization by numerical quadrature, the first Born approximation yields a superposition of fundamental solutions with complex scattering amplitudes α_q.

Point-based model of measurements

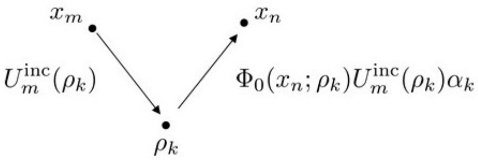
- We introduce a mesh over an imaging window with grid points $\boldsymbol{\rho}_k$ for $k = 1, \cdots, K$.
- Each grid point is a "secondary point-source" with corresponding complex amplitude, α_k.
- When the secondary point-source is inside the object, $\alpha_k \neq 0$. Otherwise, $\alpha_k = 0$.
- The (sparse) K-vector $\mathbf{x}$ whose components are α_k for $k = 1, \cdots, K$ indicate the location and shape of the objects.



Modeling the data

Let $U_m^{\text{inc}}(\rho_k)$ denote the mth incident field evaluated at ρ_k.

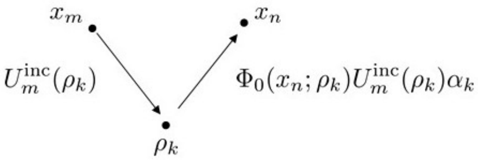
$\Phi_0(x_n;\rho_k)U_m^{\text{inc}}(\rho_k)\alpha_k$ is the field radiating from ρ_k that is measured at x_n.



Modeling the data

Let $U_m^{\text{inc}}(\rho_k)$ denote the mth incident field evaluated at ρ_k.

$\Phi_0(x_n;\rho_k)U_m^{\text{inc}}(\rho_k)\alpha_k$ is the field radiating from ρ_k that is measured at x_n.



The nth column of the data matrix B is given by

$$\mathbf{b}_n = \underbrace{\begin{bmatrix} U_1^{\text{inc}}(\rho_1) & \cdots & U_1^{\text{inc}}(\rho_K) \\ \vdots & \ddots & \vdots \\ U_M^{\text{inc}}(\rho_1) & \cdots & U_M^{\text{inc}}(\rho_K) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Phi_0(x_n;\rho_1) & & \\ & \ddots & \\ & & \Phi_0(x_n;\rho_K) \end{bmatrix}}_{\Lambda_n} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{bmatrix}}_{\boldsymbol{\alpha}}.$$

Modeling the data

Using this model, we have

$$B = A[\Lambda_1 \boldsymbol{\alpha} | \Lambda_2 \boldsymbol{\alpha} | \cdots | \Lambda_N \boldsymbol{\alpha}].$$

- According to this model, the data matrix B is given by linear combinations of the columns of the $M \times K$ matrix, A.
- The particular columns of A that are used in B are set by the non-zero components of the vector $\boldsymbol{\alpha}$.
- The non-zero components of α correspond to the grid points inside of the objects.
- It is therefore sufficient to determine which columns of A are used in B to determine the location and shape of the objects.

Multiple signal classification (MUSIC)

We compute the singular value decomposition (SVD) of B to obtain

$$B = A\,[\Lambda_1|\Lambda_2|\cdots|\Lambda_N]\,\boldsymbol{\alpha} = U\Sigma V^*.$$

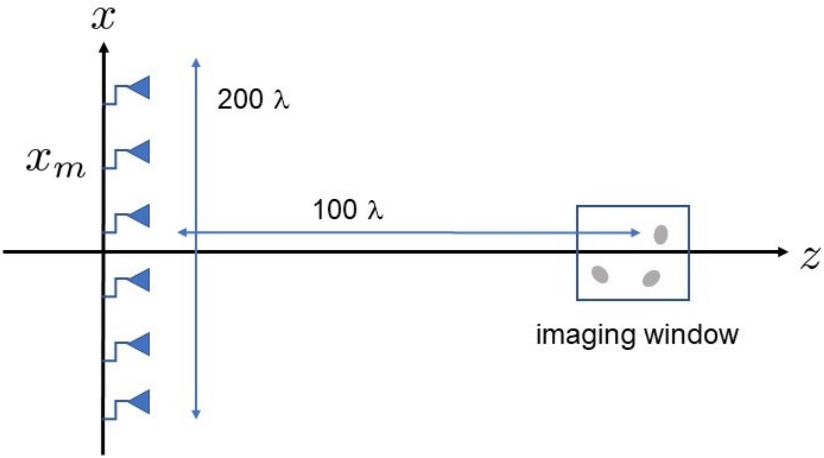
- The columns of U that correspond to first r significant singular values form an orthogonal basis for the span of the columns of A, which is called the signal subspace.
- Let $\tilde{U} = U(:, 1:r)$. Then $P = I - \tilde{U}\tilde{U}^*$ is the orthogonal projection matrix to the signal subspace.
- Let $\mathbf{a}_k$ denote the kth column of A. Then $\eta_k = \|P\mathbf{a}_k\|/\|\mathbf{a}_k\|$ gives the fraction of the column of $\mathbf{a}_k$ that is <u>not</u> in the signal subspace.
- A plot of $1/\eta_k$ for each grid point, $\boldsymbol{\rho}_k$ for $k = 1, \cdots, K$, produces an image that will show the location and shape of the objects.

MUSIC imaging algorithm

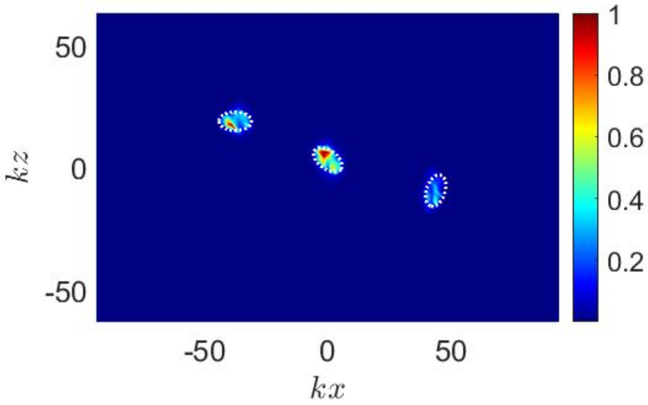
1. Compute
$$[\mathtt{U}, \mathtt{S}, \mathtt{V}] = \mathtt{svd}(\mathtt{B}).$$
2. Determine the first $\mathtt{r}$ significant singular values, and then compute
$$\mathtt{P} = \mathtt{eye}(\mathtt{M}) - \mathtt{U}(:, 1:\mathtt{r})\mathtt{U}(:, 1:\mathtt{r})'.$$
3. Given mesh of an imaging window with grid points $\boldsymbol{\rho}_k$ for $k = 1, \cdots, K$, we compute
$$\mathtt{a\_k} = \mathtt{u\_inc}(\mathtt{rho\_k}; \mathtt{x\_m})$$
and then compute
$$\mathtt{eta}(\mathtt{k}) = \mathtt{norm}(\mathtt{P} * \mathtt{a\_k})/\mathtt{norm}(\mathtt{a\_k}).$$
4. Plot $\mathtt{min}(\mathtt{eta})./\mathtt{eta}$.

Simulations

We consider a 2.5 GHz array imaging system.

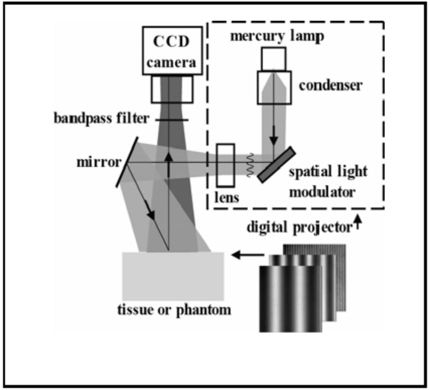


Array imaging results



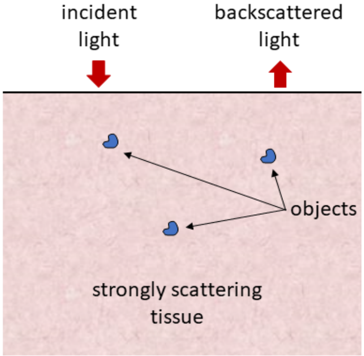
Imaging with spatially modulated light

- Introduced by Cuccia *et al.* (2005) as a means for imaging tissues.
- Projects Fourier patterns of light onto the tissue sample.
- Images in the spatial frequency domain.
- Intuitively, the higher spatial frequencies have a shorter penetration depth than lower spatial frequencies because tissues act as a low-pass filter.



Gioux, Mazhar, and Cuccia (2019)

Diffusion of spatially modulated light in tissues



S. Rohde and ADK (2017) show that the diffuse reflectance is given by

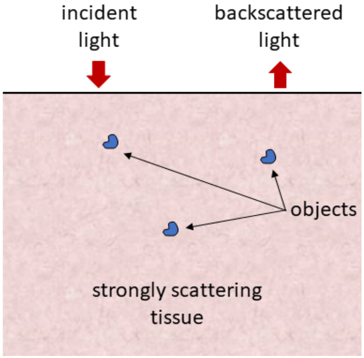
$$R_m(x_n) \sim c_0(I_0 + I_1 e^{\mathrm{i}2\pi f_m x_n}) + c_1 3D\partial_z U(x_n),$$

with U satisfying the diffusion approximation

$$-\nabla \cdot (D\nabla U) + \mu_a U = 0 \quad \text{in } z > 0,$$
$$U = I_0 + I_1 e^{\mathrm{i}2\pi f_m x} \quad \text{on } z = 0,$$

with c_0 and c_1 determined using asymptotic analysis.

Diffusion of spatially modulated light in tissues



Upon solution of the following boundary value problem,

$$-\nabla \cdot (D\nabla U) + \mu_a U = 0 \quad \text{in } z > 0,$$
$$U = I_0 + I_1 e^{i2\pi f_m x} \quad \text{on } z = 0,$$

we assume we can isolate

$$\tilde{R}_m(x_n) = -D\partial_z U(x_n, 0).$$

in measurements.

Scattering and absorbing objects have different values for the diffusion coefficient, D and the absorption coefficient, μ_a, respectively.

Measurement matrix

Suppose we illuminate the medium with M spatial frequencies, f_m for $m = 1, \cdots, M$ and measure the diffuse reflectance R at N locations, x_n for $n = 1, \cdots, N$.

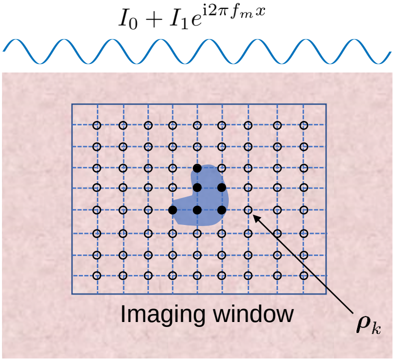
Let $R_m(x_n)$ denote the *complex* diffuse reflectance (in phase and quadrature components) for spatial frequency f_m at detector location x_n.

We organize our measurements as the following $M \times N$ matrix,

$$B = \begin{bmatrix} R_1(x_1) & \cdots & R_1(x_N) \\ R_2(x_1) & \cdots & R_2(x_N) \\ \vdots & \ddots & \vdots \\ R_M(x_1) & \cdots & R_M(x_N) \end{bmatrix}.$$

We seek to recover the location and shape of the objects from the matrix B.

Point-based model of measurements



- We introduce a mesh over an imaging window with grid points $\boldsymbol{\rho}_k$ for $k = 1, \cdots, K$.
- Each grid point is a "secondary point-source" with corresponding yield, ϕ_k.
- When the secondary point-source is inside the object, $\phi_k \neq 0$. Otherwise, $\phi_k = 0$.

Point-based model of measurements

Using the same ideas we used for waves, we find that the nth column of the data matrix B is given by

$$\mathbf{b}_n = \begin{bmatrix} R_1(x_n) \\ \vdots \\ R_M(x_m) \end{bmatrix} = \underbrace{\begin{bmatrix} U_1(\boldsymbol{\rho}_1) & \cdots & U_1(\boldsymbol{\rho}_K) \\ \vdots & \ddots & \vdots \\ U_M(\boldsymbol{\rho}_1) & \cdots & U_M(\boldsymbol{\rho}_K) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Phi_0(x_n;\boldsymbol{\rho}_1) & & \\ & \ddots & \\ & & \Phi_0(x_n;\boldsymbol{\rho}_K) \end{bmatrix}}_{\Lambda_n} \underbrace{\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_K \end{bmatrix}}_{\boldsymbol{\phi}}.$$

The model for the data matrix B is

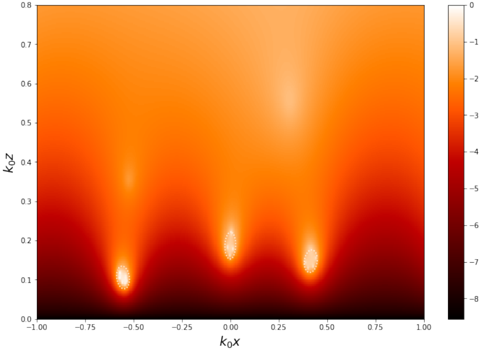
$$B = A\left[\Lambda_1\boldsymbol{\phi}|\Lambda_2\boldsymbol{\phi}|\cdots|\Lambda_N\boldsymbol{\phi}\right] = A\left[\Lambda_1|\Lambda_2|\cdots|\Lambda_N\right]\boldsymbol{\phi}.$$

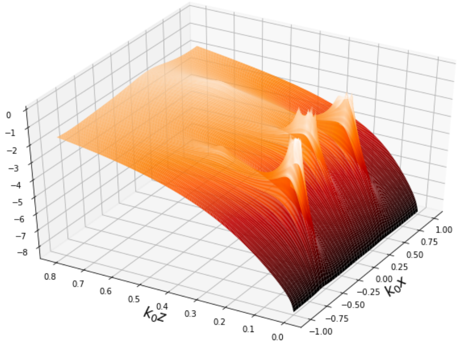
This is *exactly* the same algebraic structure as before! We use MUSIC for the inverse scattering problem.

Simulations

- Background optical properties are $\mu_s' = 1\,\text{mm}^{-1}$ and $\mu_a = 0.01\,\text{mm}^{-1}$, so that $k_0 = \sqrt{3\mu_a\mu_s'} = 0.1732\,\text{mm}^{-1}$.
- We illuminate using 100 spatial frequencies (more than usual).
- There are 501 detectors/pixels.

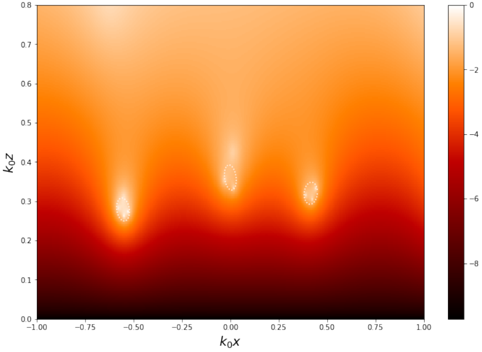
Numerical results: scattering and absorbing objects

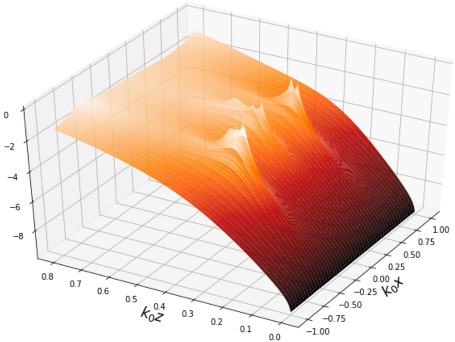




Left and right objects are scattering perturbations and center object is an absorbing object.

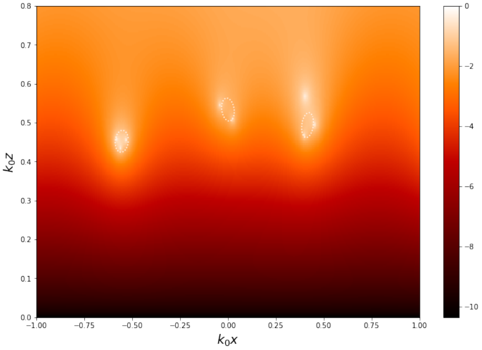
Numerical results: scattering and absorbing objects

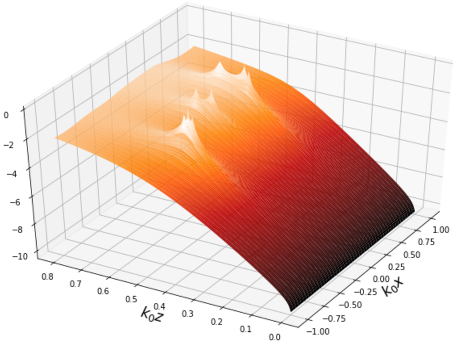




Left and right objects are scattering perturbations and center object is an absorbing object.

Numerical results: scattering and absorbing objects





Left and right objects are scattering perturbations and center object is an absorbing object.

Conclusions

- The Method of Fundamental Solutions (MFS) is effective for solving the direct scattering problem for extended objects.
- MUSIC is a simple and effective method to recover the location and support of extended objects.
- For diffusive waves, MUSIC is depth-limited.
- We are currently studying methods that seek to improve the performance of MUSIC for a broad variety of applications.