Asymptotic approximation of Boundary Integral Equations with high curvature

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Outline

1. Motivation
2. Interior Dirchlet Laplace
3. Extension to other problems
4. Conclusion
Outline

1 Motivation

2 Interior Dirchlet Laplace

3 Extension to other problems

4 Conclusion
Motivation

- Slender Body Theory [1]
  - How light is affected when obstacles have a region of high curvature.

- Wave problem : Exterior Helmholtz
- Focus on 2D in Laplace
Outline

1. Motivation
2. Interior Dirchlet Laplace
3. Extension to other problems
4. Conclusion
Introduction

Interior Dirichlet Laplace problem:

\[ \Delta u = 0 \quad \text{in} \quad D \]
\[ u = f \quad \text{on} \quad B \]

The solution is represented as a double-layer potential

\[ u(x) = \int_B \frac{\partial G}{\partial n_y}(x, y) \mu(y) d\sigma_y, \quad x \in D \]

Green’s function \( G(x, y) = -\frac{1}{2\pi} \log|x - y| \)

\( \mu \) is continuous density that satisfies the BIE

\[ -\frac{1}{2} \mu(x) + \int_B \frac{\partial G}{\partial n_y}(x, y) \mu(y) d\sigma_y = f(x), \quad x \in B \]

where \( \frac{\partial G}{\partial n_y}(x, y) = -\frac{1}{2\pi n_y} \cdot \frac{x - y}{|x - y|^2} \)
Why do we use Boundary Integral method?

+ Reduction of 1 dimension
+ High-order methods available [2]
  - Dense matrices

Challenge:

- We focus on the boundary integral equation

\[- \frac{1}{2} \mu(x) + \int_B \frac{\partial G}{\partial n_y}(x, y) \mu(y) d\sigma_y = f(x), \quad x \in B\]

for a region of high curvature.

- We have a parameter $\epsilon$ which perturbs our ellipse to have the following behavior
Solving the BIE

To solve the boundary integral equation we take advantage of parameterization

\[ y(s) = \langle \epsilon \cos s, \sin s \rangle, \quad 0 \leq s \leq 2\pi \]

Plugging it into the boundary integral equation we get

\[-\frac{1}{2} \mu(s) + \frac{1}{2\pi} \int_0^{2\pi} n_y(t) \cdot \frac{y(s) - y(t)}{|y(s) - y(t)|^2} |y'(t)| \mu(t) \, dt = f(s), \quad 0 \leq s \leq 2\pi\]

The boundary integral equation can be rewritten as

\[-\frac{1}{2} \mu(s) + \frac{1}{2\pi} \int_0^{2\pi} K(s,t; \epsilon) \mu(s) \, dt = f(s), \quad 0 \leq s \leq 2\pi\]

with kernel

\[ K(s,t; \epsilon) = \frac{\epsilon}{-1 - \epsilon^2 - (1 - \epsilon^2) \cos(s + t)} \]
Taking a closer look at the kernel

$$K(s, t; \varepsilon) = \frac{\varepsilon}{-1 - \varepsilon^2 - (1 - \varepsilon^2)\cos(s + t)}$$

Notice that $K$ is nearly singular [3] when $s + t = \pi$ or $s + t = 3\pi$ which is a challenge.

When $\cos(s + t) = -1$ then

$$K(s, t; \varepsilon) = -\frac{1}{2\varepsilon}$$

$$\lim_{\varepsilon \to 0} K(s, t; \varepsilon) \to -\infty$$
Recall the BIE:

$$-\frac{1}{2}\mu(s) + \frac{1}{2\pi} \int_0^{2\pi} K(s, t; \varepsilon) \mu(t) \, dt = f(s), \quad 0 \leq s \leq 2\pi$$

**The Periodic Trapezoid Rule**

$$-\frac{1}{2}\mu(s_i) + \frac{1}{N} \sum_{j=1}^{N} K(s_i, t_j) \mu(t_j) = f(s_i), \quad i = 1, \ldots, N$$

where \( s_i = \frac{2\pi i}{N} \) and \( t_j = \frac{2\pi j}{N} \)

We solve the System \((-\frac{1}{2}I_N + P)\mu_N = f_N\)
where \(P\) is the matrix obtained from the Periodic Trapezoid rule.
Recall the BIE:

\[-\frac{1}{2} \mu(s) + \frac{1}{2\pi} \int_{0}^{2\pi} K(s, t; \epsilon) \mu(s) dt = f(s)\]

Gauss’ Law \[3\]

\[
\int_{B} \frac{\partial G}{\partial n_y}(x, y) \mu(y) d\sigma_y = \begin{cases}
-1, & x \in D \\
-\frac{1}{2}, & x \in B \\
0, & x \in E
\end{cases}
\]

Example: We run a test case where \(f = 1\). Since \(f\) is constant and we assume \(\mu\) to be constant then Gauss’ Law will give us the following

\[-\frac{1}{2} \mu(s) - \frac{1}{2} \mu(s) = 1\]

\[\mu(s) = -1\]

The test case solution to the BIE will be \(\mu = -1\).
We run PTR for $N = 32$ quadrature points and different values of $\varepsilon$.

- PTR approximation is way off $\mu(s) \neq -1$, which is a problem.
The following is a log-log plot of the error vs. N for the periodic trapezoid rule with a fixed $\varepsilon = 0.001$
Reason:

Recall, \( K(s, t; \epsilon) = -\frac{1}{2\epsilon} \) when \( \cos(s + t) = -1 \)

then

\[
K(s, t; \epsilon) = -\frac{1}{2\epsilon}, \quad \lim_{\epsilon \to 0} K(s, t; \epsilon) \to -\infty
\]

(a) Plot of kernel for different values of \( \epsilon \).

(b) Heat map of matrix P for PTR.
Inner Asymptotic expansion

We provide an alternative numerical method for the points $y(s + t) = y(\pi)$ and $y(s + t) = y(3\pi)$. 
Inner Asymptotic expansion

We provide an alternative numerical method for the points $y(s + t) = y(\pi)$ and $y(s + t) = y(3\pi)$.

- The expression of the integral for a neighborhood around the point $y(\pi - s)$

$$I_1 = \int_{\pi - s - \Delta t}^{\pi - s + \Delta t} K(s, t; \varepsilon) \mu(t) \, dt$$
Substitutions:

1. \( t = \tau + \pi - s \) with
   \[
   dt = d\tau \quad \longrightarrow \quad I_1 = \frac{1}{2\pi} \int_{-\Delta t}^{\Delta t} K(s, \tau + \pi - s; \varepsilon) \mu(\tau + \pi - s) d\tau
   \]

2. \( \tau = \varepsilon T \) with
   \[
   d\tau = \varepsilon dT \quad \longrightarrow \quad I_1 = \frac{1}{2\pi} \int_{-\Delta T}^{\Delta T} K(s, \varepsilon T + \pi - s; \varepsilon) \mu(\varepsilon T + \pi - s) \varepsilon dT
   \]

Expanding about \( \varepsilon = 0 \):

\[
K(s, \varepsilon T + \pi - s; \varepsilon) = \frac{1}{(-2 - \frac{T^2}{2})\varepsilon} + \frac{(-12T^2 - T^4)\varepsilon}{6(4 + T^2)^2} + O(\varepsilon^3)
\]

\[
\mu(\varepsilon T + \pi - s) = \mu(\pi - s) + \varepsilon T \mu(\pi - s) + O(\varepsilon^2)
\]

\[
I_1 \sim \frac{1}{2\pi} \int_{-\Delta T}^{\Delta T} \frac{\mu(\pi - s)}{-2 - \frac{T^2}{2}} \varepsilon dT + O(\varepsilon) \sim -\arctan\left(\frac{\Delta T}{2\varepsilon}\right) \mu(\pi - s) \pi
\]
**Modified numerical method:**

At the points $y(s + t) = y(\pi)$ and $y(s + t) = y(3\pi)$ we replace PTR with the asymptotic.

\[-\frac{1}{2} \mu(s_i) + \frac{1}{N} \sum_{j=1}^{N} K(s_i, t_j) \mu(t_j) = f(s_i), \quad i = 1, \ldots, N\]

Modified BIE

\[-\frac{1}{2} \mu(s_i) + \frac{1}{N} \sum_{j=1}^{N} K(s_i, t_j) \mu(t_j) + \sum_{s_i + t_j = \pi, \ s_i + t_j \neq 3\pi} \left( - \frac{\arctan(\frac{\Delta t}{2\varepsilon}) \mu(\pi - s)}{\pi} \right) = f(s_i), \quad i = 1, \ldots, N\]

We solve the **System**

\[(-\frac{1}{2}I_N + P_m)\mu_N = f_N\]

where $P_m$ is the Matrix of the MPTR.
Numerical Results: we run the MTR for $N = 32, f = 1$ and different values of $\varepsilon$ (test case $\mu = -1$).
Comparing the Error

**Modified trapezoid rule**
- The Method does not do well when $\Delta t < \epsilon$.

**Trapezoid rule**
- Requires more points to get a good approximation of $\mu$.
Spectral approximation of the BIE:

**Fourier Series**

Recall that the boundary integral equation

\[-\frac{1}{2} \mu(s) + \frac{1}{2\pi} \int_0^{2\pi} K(s, t; \varepsilon) \mu(t) dt = f(s), \quad 0 \leq s \leq 2\pi\]

**Steps:**

1. Substitute

\[
\mu(s) = \sum_{n=-\infty}^{\infty} \hat{\mu}_n e^{ins}, \quad K(s, t; \varepsilon) = \sum_{m=-\infty}^{\infty} \hat{k}_m e^{im(s+t)}, \quad \text{and} \quad f(s) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{ins}
\]
Spectral approximation of the BIE:

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The **Fourier representation of the BIE** is given by

\[-\frac{1}{2} \hat{\mu}_n + \hat{\mu}_{-n} \hat{k}_n = \hat{f}_n, \quad n = -\infty, \ldots, \infty\]
Finding the Fourier coefficients

1. We rewrite $K(s, t; \varepsilon)$ as a rational trigonometric function [4].

$$K(s, t; \varepsilon) = \left(\frac{-\varepsilon}{1+\varepsilon^2}\right)\left(\frac{1}{1+\frac{\varepsilon^2}{1+\varepsilon^2}\cos(s+t)}\right)$$
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2. Let $c_0 = \frac{-\varepsilon}{1+\varepsilon^2}$ and $c_1 = \frac{1-\varepsilon^2}{1+\varepsilon^2}$

   with $c_1 < 1$
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2. Let $c_0 = \frac{-\varepsilon}{1+\varepsilon^2}$ and $c_1 = \frac{1-\varepsilon^2}{1+\varepsilon^2}$ with $c_1 < 1$

\[ K(s, t; \varepsilon) = \left( \frac{c_0}{1+c_1 \cos(s+t)} \right) \]

3. We obtain

\[ \hat{k}_n = c_0 \frac{1+\rho^2}{1-\rho^2} \rho^{\left| n \right|}, \quad \forall n. \]

with $I_{n,1} = \rho^n \frac{1+\rho^2}{1-\rho^2}$ and $\rho = \frac{\sqrt{1-c_1^2} - 1}{c_1}$
Finding the Fourier coefficients

1. We rewrite $K(s, t; \varepsilon)$ as a rational trigonometric function [4].

$$K(s, t; \varepsilon) = \left(\frac{-\varepsilon}{1+\varepsilon^2}\right) \left(\frac{1}{1+1+\varepsilon^2\cos(s+t)}\right)$$

2. Let $c_0 = \frac{-\varepsilon}{1+\varepsilon^2}$ and $c_1 = \frac{1-\varepsilon^2}{1+\varepsilon^2}$ with $c_1 < 1$

$$K(s, t; \varepsilon) = \left(\frac{c_0}{1+c_1\cos(s+t)}\right)$$

3. We obtain $\hat{k}_n = c_0 \frac{1+\rho^2}{1-\rho^2} \rho^{|n|} \quad \forall n$

with $I_{n,1} = \rho^n \frac{1+\rho^2}{1-\rho^2} \quad n \geq 0$

and $\rho = \sqrt{1-c_1^2-1}$

The truncated system becomes

$$-\frac{1}{2} \hat{\mu}_n + \hat{\mu}_{-n} \hat{k}_n = \hat{f}_n, \quad n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$$
FFT Results

\textbf{Figure 3} – Plot of the error VS. $\epsilon$ of Spectral method for different values of epsilon.
**Summary - Interior Dirchlet Laplace**

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Extension of the methods to
- Exterior Neumann Laplace
## Summary - Interior Dirichlet Laplace

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- Extension of the methods to
  - Exterior Neumann Laplace
  - Scattering Problem
Outline

1 Motivation

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Exterior Neumann

Exterior Neumann Laplace problem:

\[ \Delta u = 0 \quad \text{in} \quad E \]
\[ \frac{\partial u}{\partial n_x} = f \quad \text{on} \quad B \]

where \( E := \mathbb{R}^2 / \bar{D} \)

The solution is represented as a single-layer potential

\[ u(x) = \int_B G(x, y) \mu(y) d\sigma_y, \quad x \in E \]

Green’s function \( G(x, y) = -\frac{1}{2\pi} \log|x - y| \)

\( \mu \) is continuous density that satisfies the BIE

\[ \frac{1}{2} \mu(x) - \int_B \frac{\partial G}{\partial n_x}(x, y) \mu(y) d\sigma_y = f(x), \quad x \in B \]
Nearly Singular Behavior

using parameterization \( y(s) = \langle \varepsilon \cos s, \sin s \rangle, \quad 0 \leq s \leq 2\pi \) We get the BIE written as

\[
\frac{1}{2} \mu(s) + \frac{1}{2\pi} \int_0^{2\pi} K(s, t; \varepsilon) \mu(s) \, dt = f(s), \quad 0 \leq s \leq 2\pi
\]

(1)

with kernel

\[
K(s, t; \varepsilon) = -\frac{\varepsilon \sqrt{1 + \varepsilon^2 - (-1 + \varepsilon^2) \cos(2t)}}{\sqrt{2(-1 - \varepsilon^2 + (-1 + \varepsilon^2) \cos(s + t))} \sqrt{\cos^2(s) + \varepsilon^2 \sin^2(s)}}
\]

Notice that \( K \) is nearly singular when \( s + t = \pi \) or \( s + t = 3\pi \). When \( \cos(s + t) = -1 \) then

\[
K(s, \pi - s; \varepsilon) = -\frac{\sqrt{1 + \varepsilon^2 - (-1 + \varepsilon^2) \cos(2(\pi - s))}}{2\sqrt{2} \varepsilon \sqrt{\cos^2(s) + \varepsilon^2 \sin^2(s)}}
\]

\[
\lim_{\varepsilon \to 0} K(s, t; \varepsilon) \to -\infty
\]
PTR system: \((\frac{1}{2}I_N + P_N)\mu_N = f_N\).

Similar to Interior Dirichlet Laplace problem:

- Inner Expansion
- Spectral Method
Inner Asymptotic Expansion:

\[ I_1 = \int_{\pi - s - \Delta t}^{\pi - s + \Delta t} K(s, t; \varepsilon) \mu(s) \, dt \sim 2 \arctan\left(\frac{\Delta t}{2\varepsilon}\right) + O(\varepsilon) \]
Challenges:

- No exact Solution
- No analytic Fourier Series Coefficients available

Spectral Method

- Do direct FFT - Requires a lot of quadrature points
Scattering Problem

Exterior Helmholtz problem:

\[ \Delta u + k^2 u = 0, \quad \text{in} \quad E \]
\[ u = f, \quad \text{on} \quad B. \]

+ radiation condition

\[ \downarrow \]

The solution is represented as a double- and single-layer potentials

\[ u(x) = \int_B \left( \frac{\partial G}{\partial n_y}(x, y) - ikG(x, y) \right) \mu(y) d\sigma_y, \quad x \in E \]

Green’s function \[ G(x, y) = \frac{i}{4}H_0^{(1)}(k|x - y|) \]
\( \mu \) is continuous density that satisfies

\[
\frac{1}{2} \mu(x) + \int_B \left( \frac{\partial G}{\partial n_y}(x,y) - ikG(x,y) \right) \mu(y) d\sigma_y = f(x), \quad x \in B
\]

**Challenges**

- Singular kernel at \( x = y \)
- \( \varepsilon \) affects the kernel

We use Kress quadrature [5] with N quadrature points to solve the system

\[
\left( \frac{1}{2} I_N + P_H \right) \mu_N = f_N
\]
Current Investigation: Inner Asymptotic Expansion
Outline

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Conclusion

- Regions of high curvature affect BIE
- Periodic Trapezoid Rule not always effective.
- To address this we do an asymptotic expansion and sometimes use Spectral method.
- Work in progress:
  - Helmholtz
  - Extension to other boundary shapes
References


Thank you!