Applied Math Preliminary Exam: Complex Analysis

University of California, Merced, January 2016

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work. Assume $z = x + iy$, where $x, y \in \mathbb{R}$ wherever relevant.

Problem 1. (5 points each)

(a) Define precisely what it means for a function $f\colon \mathbb{C} \to \mathbb{C}$ to be analytic at z_0.

(b) Write $\dfrac{4+4\sqrt{3}i}{1-\sqrt{3}i}$ in polar notation, $re^{i\theta}$, where $r \geq 0$ and $-\pi < \theta \leq \pi$.

(c) Show that for $z \in \mathbb{C}$, $|z^2| = |z|^2$.

(d) Find all the values of $(4i)^{1/4}$.

Problem 2. (5 points each)

(a) Give an example of a complex function that is differentiable everywhere on the complex plane and is bounded. Explain.

(b) Show that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ for $z_1, z_2 \in \mathbb{C}$.

(c) Let n be an integer. Compute

$$\int_0^{2\pi} e^{in\theta} d\theta.$$

Problem 3. (20 points total)

(a) (5 points) Find and identify the kind of singularities (pole, removable, or essential) of the function

$$f(z) = \frac{(1-\cos z)e^{1/(z-2)}}{z^2(z+2)^2}$$

(b) (15 points) Find the Laurent series for

$$f(z) = \frac{1}{(z+4)(z+2)}$$

valid for $2 < |z| < 4$.

Problem 4. (15 points) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2} dx.$$

Problem 5. (5 points each)

(a) Find a fractional linear transformation that maps 0 to i, 1 to 2 and -1 to 4.

(b) Describe mathematically and draw what happens to the line $y = x + 1$ under the mapping $f(z) = 1/z$.

(c) Describe mathematically and draw what happens to the half-disc $D = \{z \in \mathbb{C}\colon \mathrm{Im}(z) \geq 0 \text{ and } |z| \leq 1\}$ under the mapping

$$f(z) = \frac{-iz + i}{z+1}$$

Problem 6. (5 points each) Short answers.

(a) Let C be a circle centered at $z = 0$ with radius 4. Compute $\displaystyle\int_C \frac{e^z}{z-1}\,dz$. Explain.

(b) Define precisely the *residue* of a function $f(z)$ at an isolated singular point z_0, and explain its significance.

(c) Find all the points where $f(z) = y + ix$ is differentiable. Also find where it is analytic.