Wave propagation over the shelf or isolated obstacles

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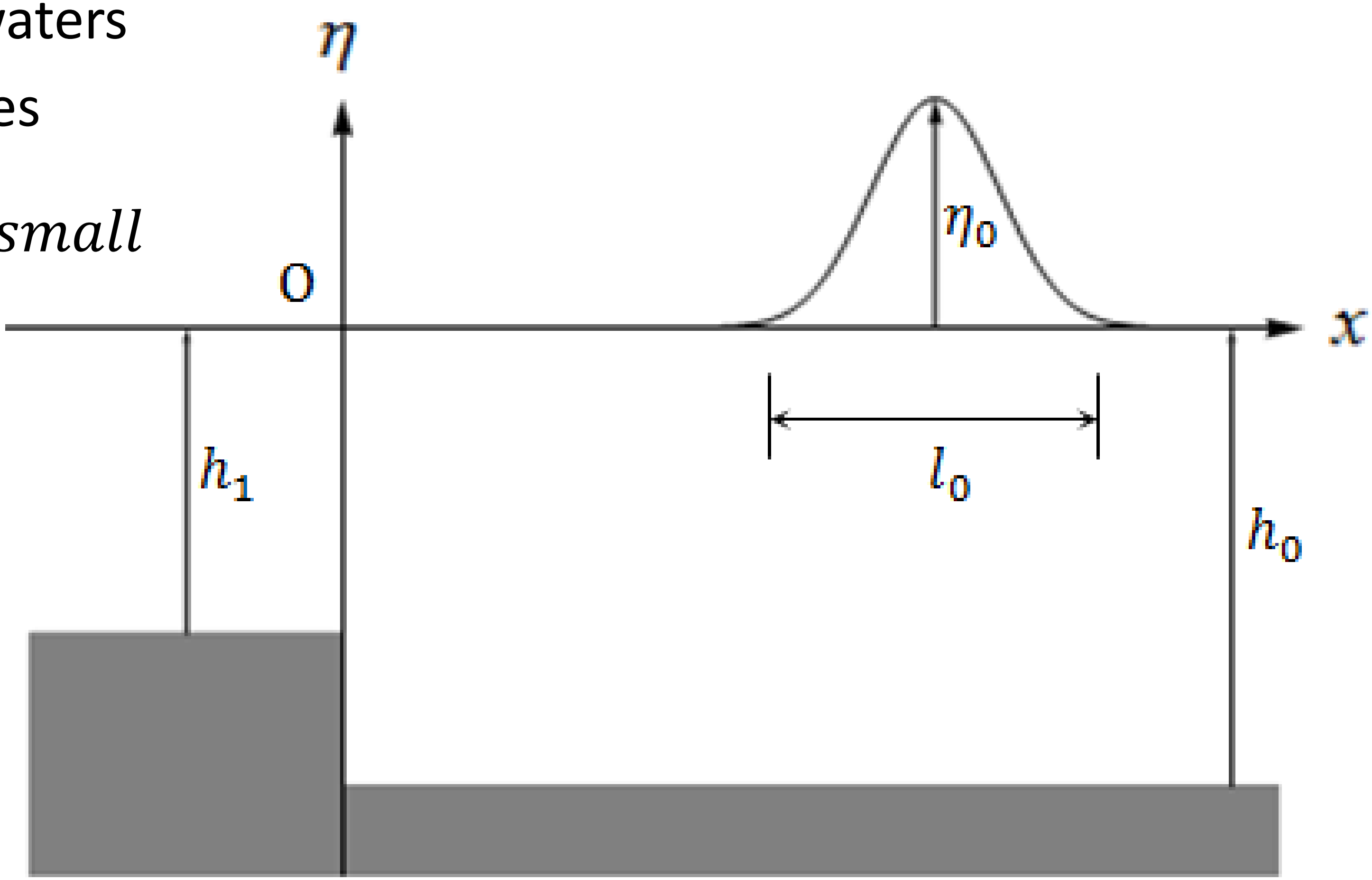
Contents

Physical System

- Shallow waters
- Long waves

$$h_0/l_0 \equiv small$$



Nonlinear Shallow-Water Equations

Let $\varphi^* = \varphi/\varphi_0, \eta^* = \eta/\eta_0, u^* = u/u_0, u_0 = \eta_0\sqrt{g/\varphi_0}, \quad \eta_0/\varphi_0 \equiv \varepsilon \ll 1$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[u(\varphi + \varepsilon\eta)\right] = 0 \longleftarrow$$ ***mass conservation***

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[\eta + \varepsilon u^2/2] = 0 \longleftarrow$$ ***momentum conservation***

initial conditions:

$$t = 0: \quad \eta(x,0) = f(x)$$
$$u(x,0) = 0$$

conditions at infinity:

$$x \to \pm\infty: \quad \eta(x,t) - f(x) = 0$$
$$u(x,t) = 0$$

$$\varphi(x) = \begin{cases} 1, & x > 0 \\ k^2, & x < 0 \end{cases}$$

conjugation at the step:

$$x = 0:$$

$$[u(\varphi + \varepsilon\eta)]_{0-}^{0+} = 0, \quad \left[\varepsilon(\varphi + \varepsilon\eta)u^2 + \frac{1}{2}\varepsilon\eta^2 + (\eta - \eta_r)\varphi\right]_{0-}^{0+} = 0$$

Linear Approximation

Linearized Equations:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}[\eta+\varepsilon u^2/2]=0$$

$$\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\big[u(\varphi+\varepsilon\eta)\big]=0$$

$$\xrightarrow{\varepsilon\to 0}$$

$$\frac{\partial u}{\partial t}=-\frac{\partial \eta}{\partial x}+O(\varepsilon)$$

$$\frac{\partial \eta}{\partial t}=-\frac{\partial}{\partial x}[\varphi u]+O(\varepsilon)$$

$x=0:$

$$\eta_R=\eta_L+O(\varepsilon)$$

$$u_R=k^2u_L+O(\varepsilon)$$

Linear Solutions:

On the shelf ($x<0$):

$$\eta_L(x,t)=\frac{1}{2}[f(x-kt)+q(x+kt)]+O(\varepsilon)$$

$$u_L(x,t)=\frac{1}{2k}[f(x-kt)-q(x+kt)]+O(\varepsilon)$$

Seafloor ($x>0$):

$$\eta_R(x,t)=\frac{1}{2}[p(x-t)+f(x+t)]+O(\varepsilon)$$

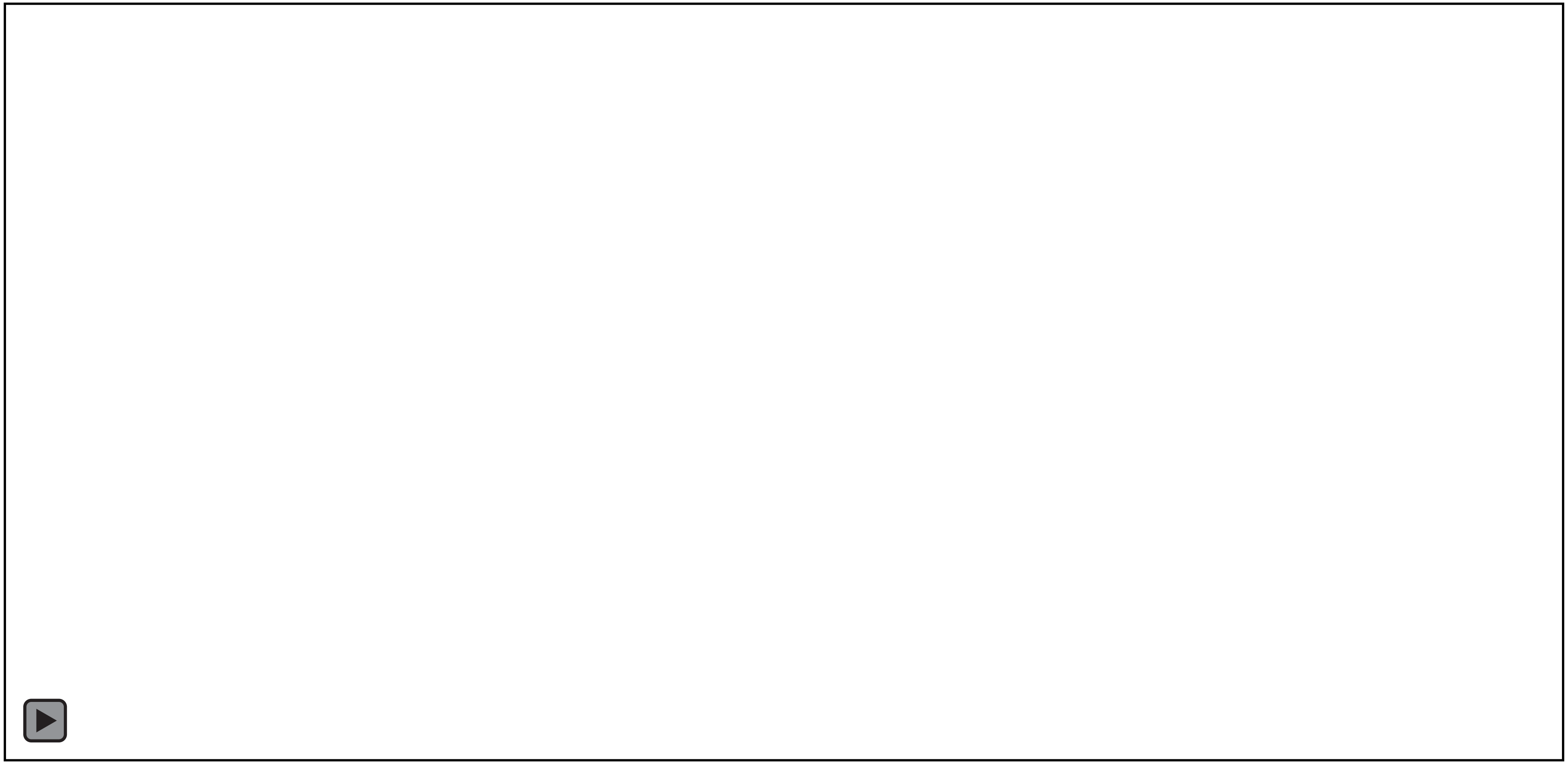
$$u_R(x,t)=\frac{1}{2}[p(x-t)-f(x+t)]+O(\varepsilon)$$

wave shapes

$$p(\xi)=\begin{cases}\dfrac{2k}{1+k}f(k\xi)+\dfrac{1-k}{1+k}f(-\xi), & \xi<0\\ f(\xi), & \xi>0\end{cases}$$ ← ***reflection off step***

$$q(y)=\begin{cases}\dfrac{2}{1+k}f(y/k)-\dfrac{1-k}{1+k}f(-y), & y>0\\ f(y), & y<0\end{cases}$$ ← ***transmission onto shelf***

Linear Propagation



Nonlinear Perturbation

linear wave → η_0, ***nonlinear correction*** → $\varepsilon\eta_1$

$$\eta \sim \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2)$$

$$u \sim u_0 + \varepsilon u_1 + O(\varepsilon^2)$$

ε⁰ equations

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + \dfrac{\partial \eta_0}{\partial x} = 0 \\[2ex] \dfrac{\partial \eta_0}{\partial t} + \dfrac{\partial}{\partial x}[\varphi u_0] = 0 \end{cases}$$

(already solved)

ε¹ equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{\partial \eta_1}{\partial x} = -u_0 \dfrac{\partial u_0}{\partial x} \\[2ex] \dfrac{\partial \eta_1}{\partial t} + \dfrac{\partial}{\partial x}[\varphi u_1] = -\dfrac{\partial \eta_0 u_0}{\partial x} \end{cases}$$

$$t = 0: \quad \eta_1(x,0) = u_1(x,0) = 0$$

$$x \to \pm\infty: \quad \eta_1(x,t) = u_1(x,t) = 0$$

(inhomogeneous equations)

Simplifications

Transformation of Unknowns

$$W_R = \eta_{R1} + u_{R0}^2/2, \qquad U_R = u_{R1} + \eta_{R0} u_{R0},$$

$$W_L = \eta_{L1} + u_{L0}^2/2, \qquad U_L = k^2 u_{L1} + \eta_{L0} u_{L0}$$

Riemann variables

$$\xi = x - t, \qquad \nu = x + t,$$

$$z = x - k\,t, \quad y = x + k\,t$$

$t = 0:$

$$\xi = \nu = y = z = x,$$

$$W_{\mathrm{R}} = W_L = U_R = U_L = 0$$

$\boldsymbol{x > 0}$:

$$\begin{cases} \dfrac{\partial}{\partial \nu}\left(W_R + U_R\right) = \dfrac{\partial}{\partial \nu}\left(\dfrac{\eta_{0r} u_{0r}}{2} + \dfrac{u_{0r}^2}{4}\right) - \dfrac{\partial}{\partial \xi}\left(\dfrac{\eta_{0r} u_{0r}}{2} + \dfrac{u_{0r}^2}{4}\right), \\ \dfrac{\partial}{\partial \xi}\left(W_R - U_R\right) = -\dfrac{\partial}{\partial \xi}\left(\dfrac{\eta_{0r} u_{0r}}{2} - \dfrac{u_{0r}^2}{4}\right) + \dfrac{\partial}{\partial \nu}\left(\dfrac{\eta_{0r} u_{0r}}{2} - \dfrac{u_{0r}^2}{4}\right), \end{cases}$$

$\boldsymbol{x < 0}$:

$$\begin{cases} \dfrac{\partial}{\partial y}\left(kW_L + U_L\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\eta_{0l} u_{0l}}{2} + k\dfrac{u_{0l}^2}{4}\right) - \dfrac{\partial}{\partial z}\left(\dfrac{\eta_{0l} u_{0l}}{2} + k\dfrac{u_{0l}^2}{4}\right), \\ \dfrac{\partial}{\partial z}\left(kW_L - U_L\right) = -\dfrac{\partial}{\partial z}\left(\dfrac{\eta_{0l} u_{0l}}{2} - k\dfrac{u_{0l}^2}{4}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\eta_{0l} u_{0l}}{2} - k\dfrac{u_{0l}^2}{4}\right), \end{cases}$$

Weakly Nonlinear Waves

$x < 0$:

$$\eta_L(z,y) \sim \eta_{L0}(z,y) + \varepsilon\,\eta_{L1}(z,y) + O(\varepsilon^2)$$

$$\eta_{L1}(z,y) = W_L(z,y) - u_{L0}^2(z,y)/2$$

$$W_R(\xi,\nu) = \begin{cases} W_{RG}(\xi,\nu); & \xi > 0 \\ W_{RG}(\xi,\nu) + \lambda(\xi)/2; & \xi < 0 \end{cases}$$

$$\lambda(\xi) = \frac{2k}{1+k}[W_{LG}(k\xi,-k\xi) + U_{LG}(k\xi,-k\xi)/k - W_{RG}(\xi,-\xi) - U_{RG}(\xi,-\xi)/k]$$

$$W_{RG}(\xi,\nu) = \frac{1}{16}[(p(\xi) - f(\nu))^2 - 3(\nu-\xi)J_{WR}(\xi,\nu) + I_{WR}(\xi,\nu)]$$

$$U_{RG}(\xi,\nu) = \frac{1}{16}[p^2(\xi) - f^2(\nu) - 3(\nu-\xi)J_{UR}(\xi,\nu) + I_{UR}(\xi,\nu)]$$

$$\nu - \xi = 2t$$

$$J_{WR}(\xi,\nu) = p(\xi)p'(\xi) - f(\nu)f'(\nu);$$

$$I_{WR}(\xi,\nu) = p'(\xi)[F(\nu) - F(\xi)] + f'(\nu)[P(\xi) - P(\nu)];$$

$$J_{UR}(\xi,\nu) = p(\xi)p'(\xi) + f(\nu)f'(\nu);$$

$$I_{UR}(\xi,\nu) = p'(\xi)[F(\nu) - F(\xi)] - f'(\nu)[P(\xi) - P(\nu)];$$

$x > 0$:

$$\eta_R(\xi,\nu) \sim \eta_{R0}(\xi,\nu) + \varepsilon\,\eta_{R1}(\xi,\nu) + O(\varepsilon^2)$$

$$\eta_{R1}(\xi,\nu) = W_R(\xi,\nu) - u_{R0}^2(\xi,\nu)/2$$

$$W_L(z,y) = \begin{cases} W_{LG}(z,y); & y < 0 \\ W_{LG}(z,y) + \psi(y)/2k; & y > 0 \end{cases}$$

$$\psi(y) = \frac{2k}{1+k}[W_{RG}(-y/k, y/k) + U_{RG}(-y/k, y/k)/k - W_{LG}(-y,y) + U_{LG}(-y,y)]$$

$$W_{LG}(z,y) = \frac{1}{16k^2}[(f(z) - q(y))^2 - 3(y-z)J_{WL}(z,y) + I_{WL}(z,y)]$$

$$U_{LG}(z,y) = \frac{1}{16k}[2(f^2(z) - q^2(y))^2 - 3(y-z)J_{UL}(z,y) + I_{UL}(z,y)]$$

$$J_{WL}(z,y) = f(z)f'(z) - q(y)q'(y);$$

$$I_{WL}(z,y) = f'(z)[Q(y) - Q(z)] + q'(y)[F(z) - F(y)];$$

$$J_{UL}(z,y) = f(z)f'(z) + q(y)q'(y);$$

$$I_{UL}(z,y) = f'(z)[Q(y) - Q(z)] - q'(y)[F(z) - F(y)];$$

$$y - z = 2kt$$

$$F(z) = \int_0^z f(x)dx; \quad P(z) = \int_0^z p(x)dx; \quad Q(z) = \int_0^z q(x)dx$$

Renormalization

Expansion of Exact Characteristics in ε

$$\xi = s_{1R} + \varepsilon \xi_R(s_{1R}, t), \quad \nu = s_{2R} + \varepsilon \nu_R(s_{2R}, t)$$

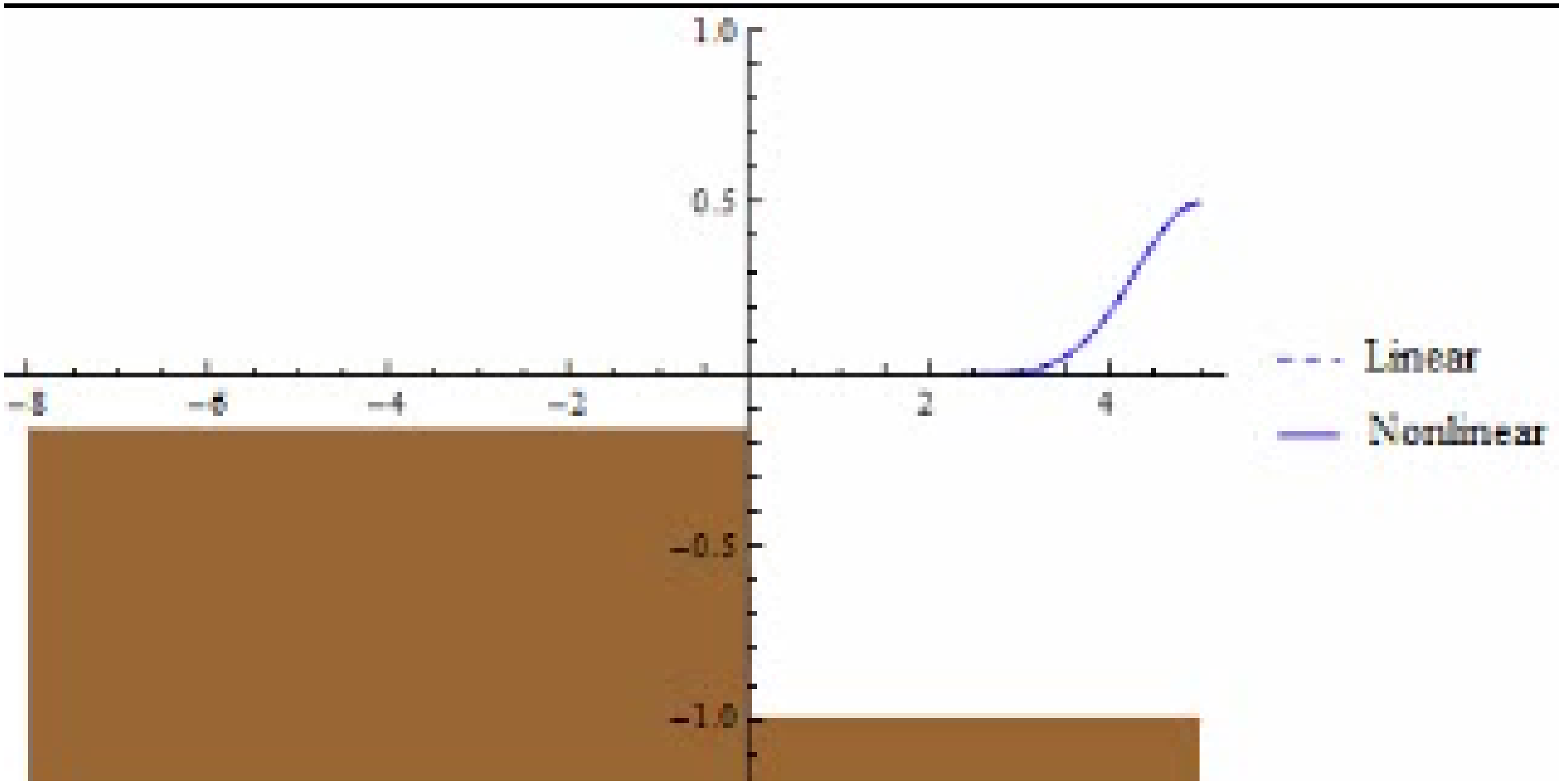
$$z = s_{1L} + \varepsilon z_L(s_{1L}, t), \quad y = s_{2L} + \varepsilon y_L(s_{2L}, t)$$

Conditions for secular terms to vanish

$$\xi_R(s_{1R}, t) = 3t\, p(s_{1R})/4, \quad \nu_R(s_{2R}, t) = -3t\, f(s_{2R})/4$$

$$z_L(s_{lL}, t) = 3\varepsilon\, f(s_{1L})t/(4k), \quad y_L(\omega_{2L}, t) = -3\varepsilon\, q(s_{2L})t/(4k).$$

Linear and Nonlinear Propagation



Riemann wave exact solutions

Below shelf $x > 0$

$$\eta_R(x,t) = \phi(x - t\,[3\sqrt{1+\varepsilon\eta_R(x,t)} - 2])$$

$$u_R(x,t) = 2(\sqrt{1+\varepsilon\eta_R(x,t)} - 1)/\varepsilon$$

Above shelf $x < 0$

$$\eta_L(x,t) = \chi(x + t\,[3\sqrt{k^2+\varepsilon\eta_L(x,t)} - 2k])$$

$$u_L(x,t) = -2[\sqrt{k^2+\varepsilon\eta_L(x,t)} - k]/\varepsilon$$

Initial splitting wave profile

$$\phi(\xi) = f(\xi)/2$$

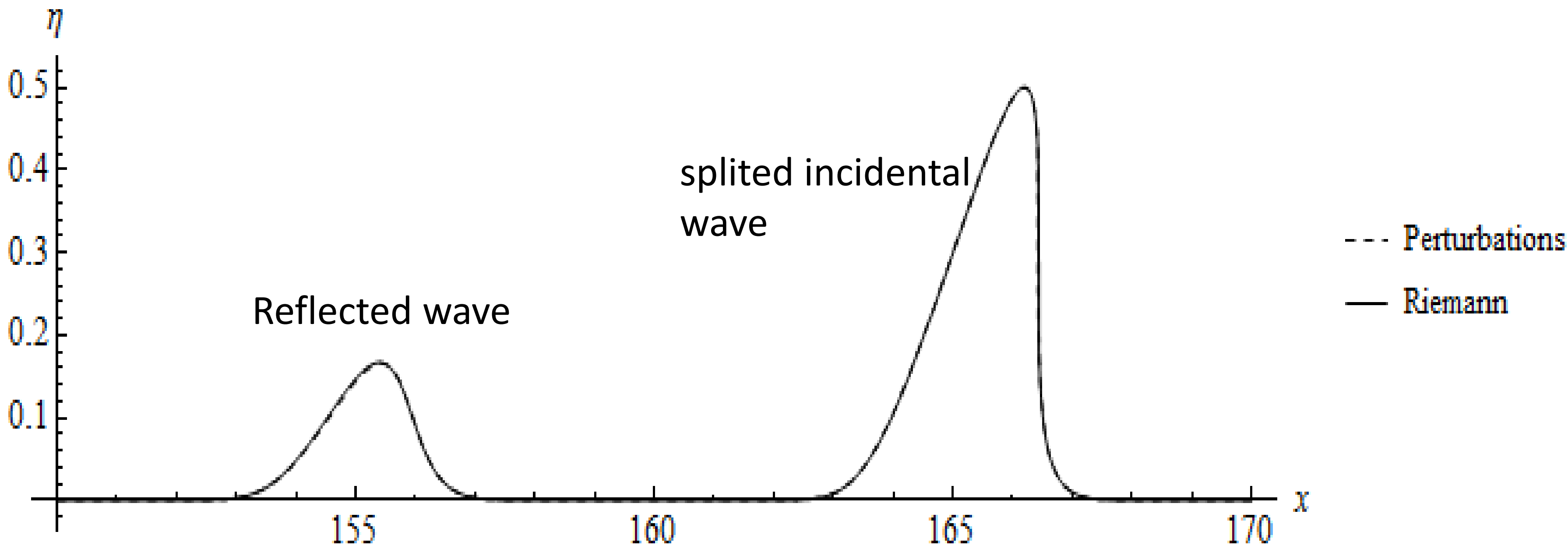
Reflected wave profile

$$\phi(\xi) = \frac{1-k}{2(1+k)} f(-\xi)$$

Transmitted wave profile

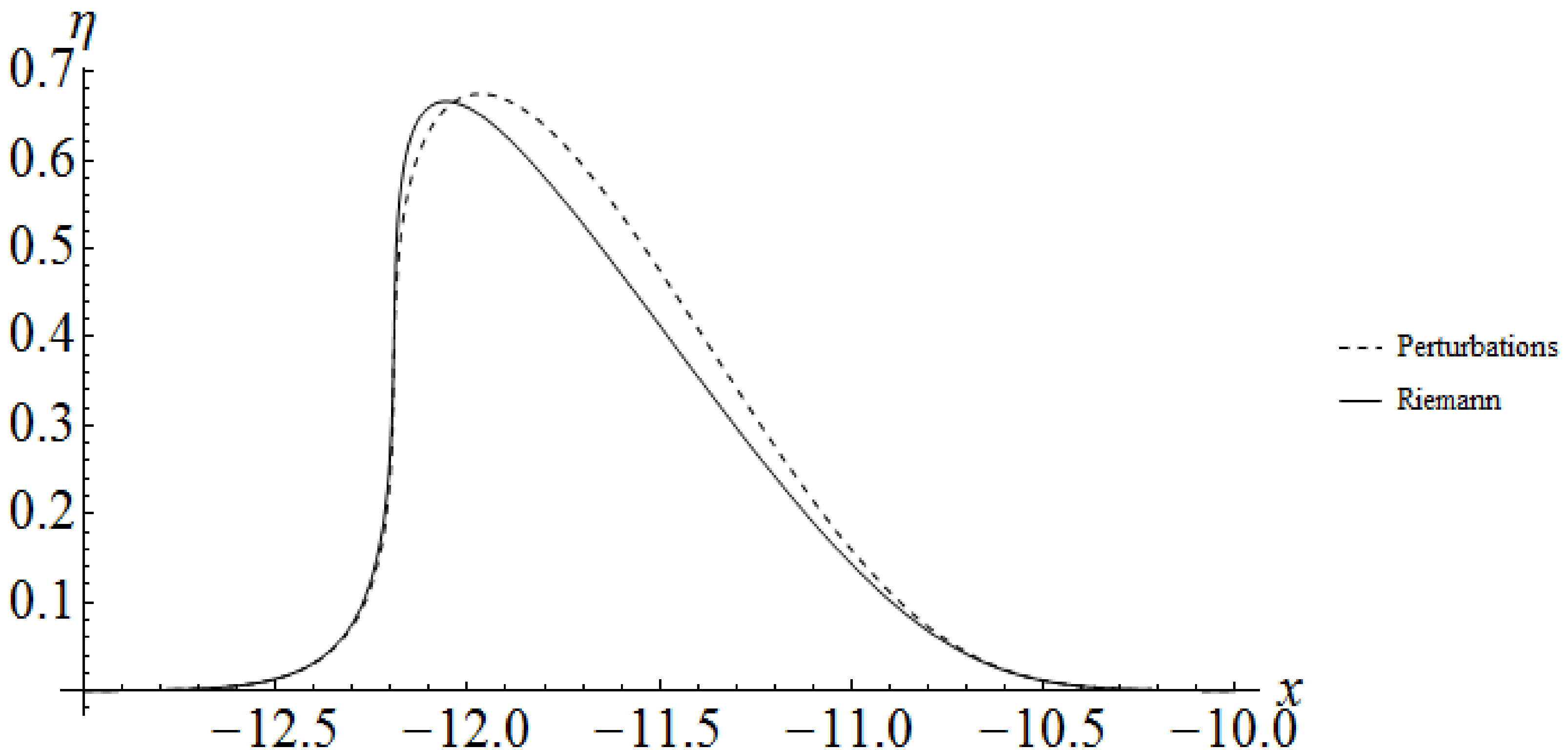
$$\chi(y) = \frac{1}{(1+k)} f\left(\frac{y}{k}\right)$$

Comparison



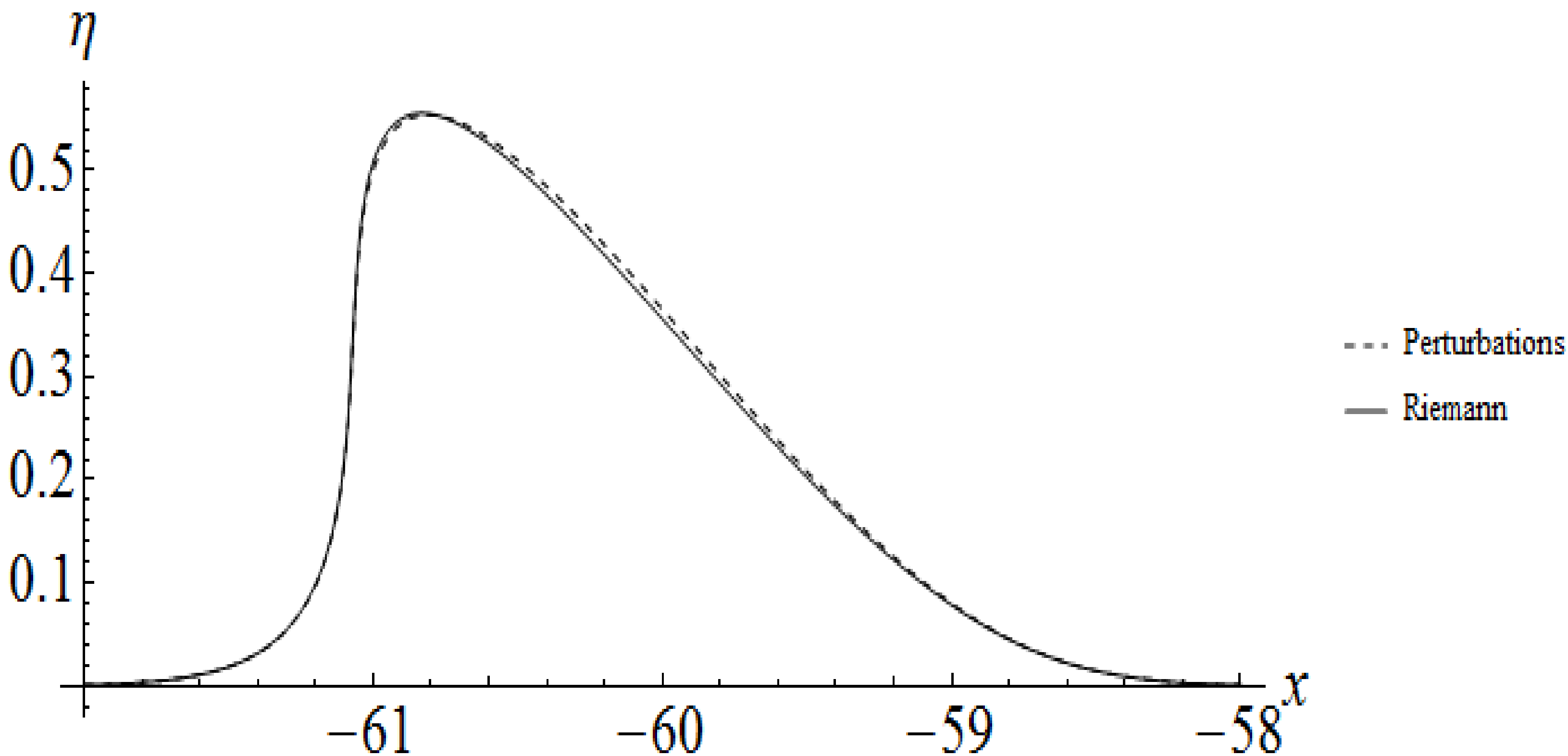
k=0.5, t=160, x_0=5, $f(x)$=Exp[-(x-x_0)^2]

Comparison



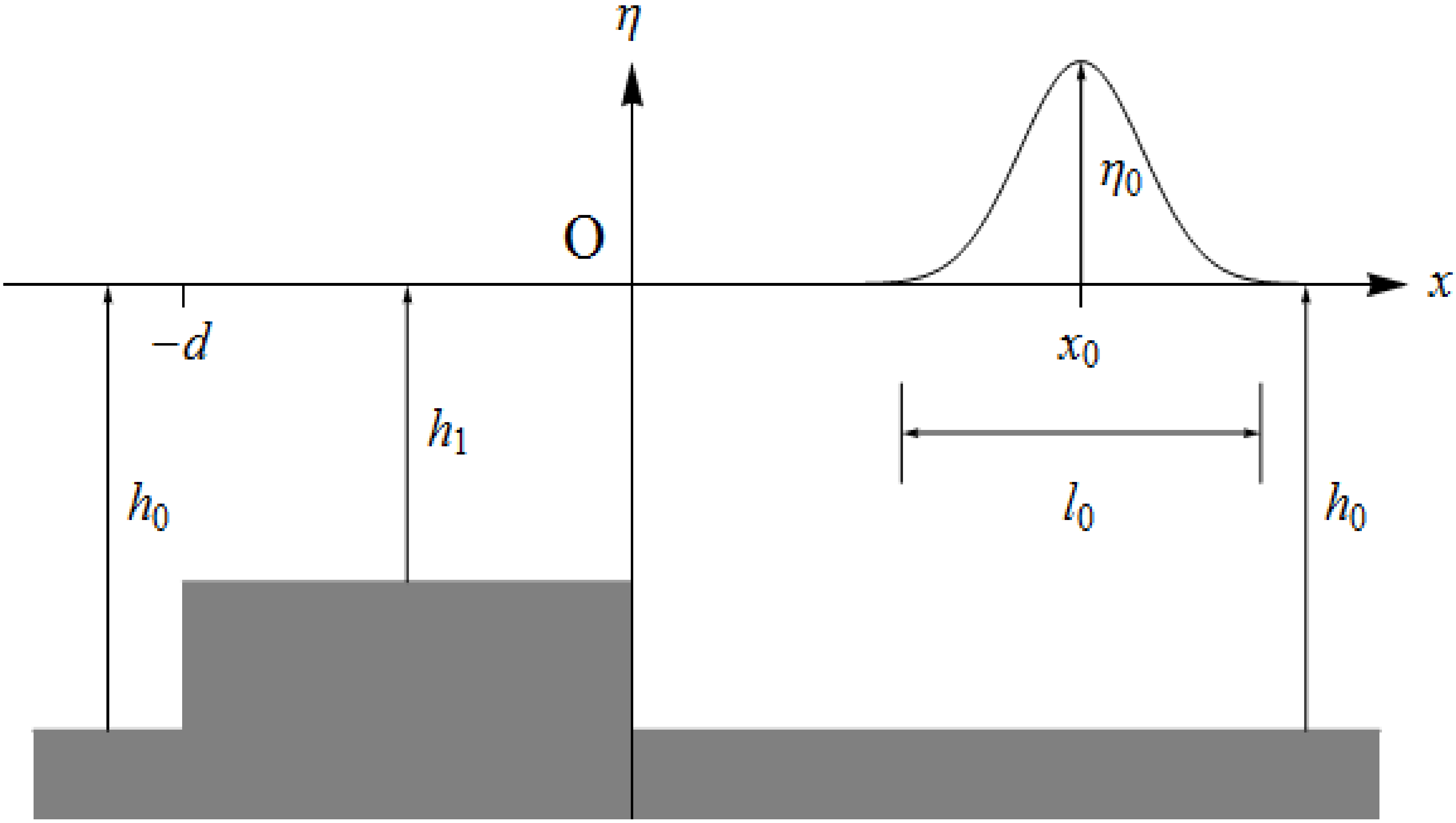
k=0.5, t=28, x_0=5, $\epsilon = 0.01$, $f(x)$=Exp[-$(x-x_0)^2$]

Comparison



k=0.8, t=28, x_0=5, $\epsilon = 0.01$, $f(x)$=Exp[-$(x-x_0)^2$]

Submerged Obstacle



Analytical Solution

Method of solution: Laplace Transforms

Above obstacle $-d < x < 0$

$\eta_M(x,t) = [f(\nu_M)\theta(\nu_M + d) + f(\xi_M)\theta(-\xi_M - d)]/2 + \left(\eta_M^{\prime,\nu}(\nu_M) + \eta_M^{\prime,\xi}(\xi_M)\right)/$
$[2(1+k)^2];$

$$\eta_M^{\prime,\nu}(\nu) = \sum_{n=0}^{\infty}[(1-k)/(1+k)]^{2n} \sum_{j=1}^{4} g_j^{M,\nu} F_j^{M,\nu}\left(h_{jn}^{M,\nu}(\nu)\right)$$
$$F_1^{M,\nu}(h) = f(h)\theta(h);\ h_{1n}^{M,\nu}(\nu) = -(\nu + 2nd + 2d)/k;$$
$$F_2^{M,\nu}(h) = f(h)\theta(-h-d);\ h_{2n}^{M,\nu}(\nu) = -h_{1n}^{M,\nu}(\nu) - d(1+k)/k;$$
$$F_3^{M,\nu}(h) = f(h)\theta(h+d)\theta(-h);\ h_{3n}^{M,\nu}(\nu) = kh_{1n}^{M,\nu}(\nu);$$
$$F_4^{M,\nu}(h) = f(h)\theta(-h-d)\theta(h);\ h_{4n}^{M,\nu}(\nu) = -kh_{1n}^{M,\nu}(\nu);$$
$$g_1^{M,\nu} = -2(1-k);\ g_2^{M,\nu} = 2(1+k);\ g_3^{M,\nu} = -(1-k^2);$$
$$g_4^{M,\nu} = -(1-k)^2;$$
$$\eta_M^{\prime,\xi}(\xi) = \sum_{n=0}^{\infty}[(1-k)/(1+k)]^{2n} \sum_{j=1}^{4} g_j^{M,\xi} F_j^{M,\xi}\left(h_{jn}^{M,\xi}(\xi)\right)$$
$$F_1^{M,\xi}(h) = f(h)\theta(-h-d);\ h_{1n}^{M,\xi}(\xi) = -(\xi - 2nd - d + kd)/k;$$
$$F_2^{M,\xi}(h) = f(h)\theta(h);\ h_{2n}^{M,\xi}(\xi) = -h_{1n}^{M,\xi}(\xi) - d + d/k;$$
$$F_3^{M,\xi}(h) = f(h)\theta(h+d)\theta(-h);\ h_{3n}^{M,\xi}(\xi) = -kh_{2n}^{M,\xi}(\xi);$$
$$F_4^{M,\xi}(h) = f(h)\theta(-h-d)\theta(h);\ h_{4n}^{M,\xi}(\xi) = kh_{2n}^{M,\xi}(\xi);$$
$$g_1^{M,\xi} = g_1^{M,\nu};\ g_2^{M,\xi} = g_2^{M,\nu};\ g_3^{M,\xi} = g_3^{M,\nu};\ g_4^{M,\xi} = -(1+k)^2,$$

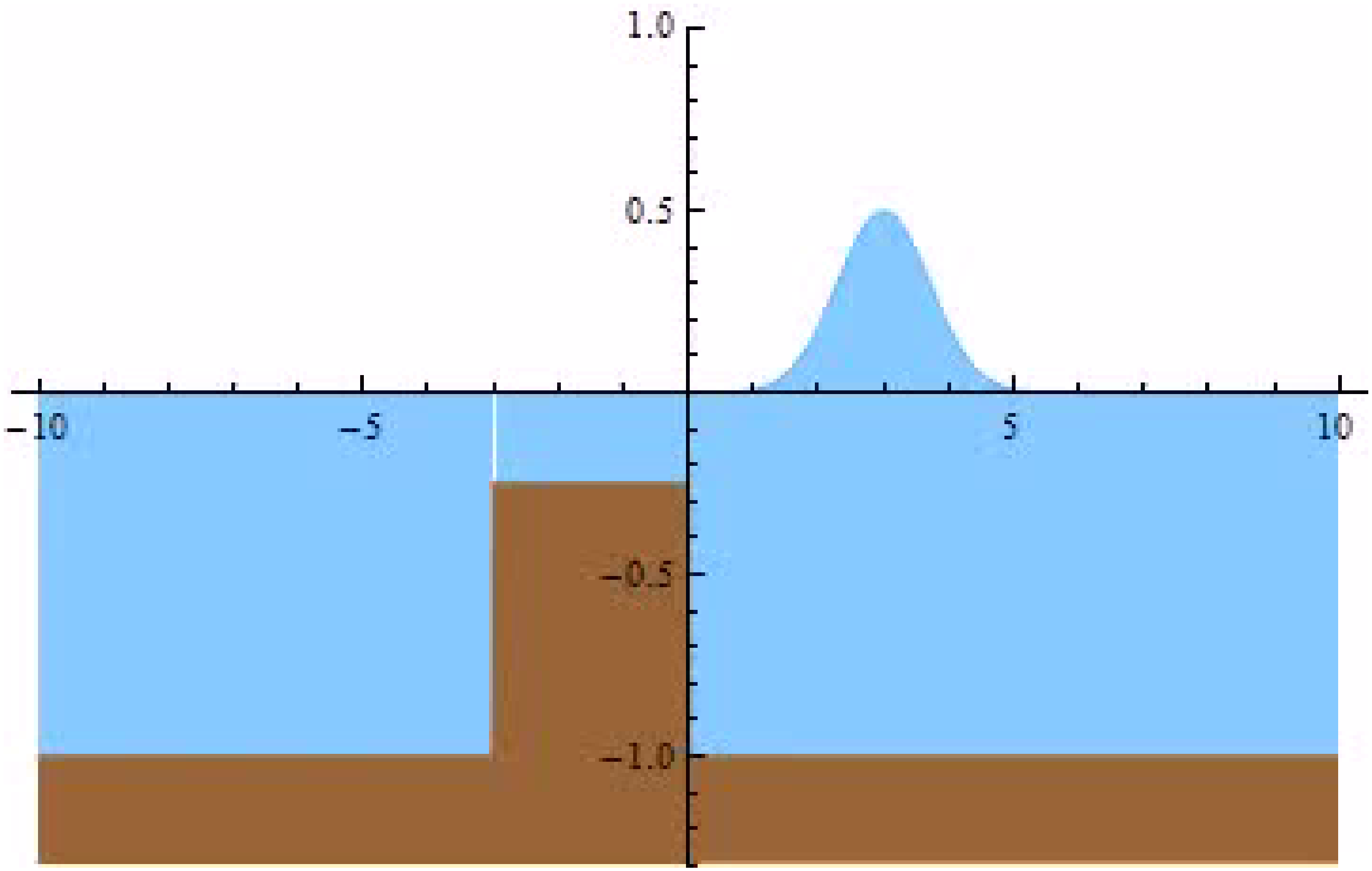
Left of obstacle $x < -d$

$$\eta_L(x,t) = [f(\nu_L) + f(\xi_L)\theta(-\xi_L - d)]/2 + \eta_L'(\xi_L)/[2(1+k)^2]; \quad (101)$$
$$\eta_L'(\xi) = \sum_{n=0}^{\infty}[(1-k)/(1+k)]^{2n} \sum_{j=1}^{5} g_j F_j^L(h_{jn}^L(\xi));$$
$$F_1^L(h) = f(h)\theta(h);\ h_{1n}^L(\xi) = \xi - d(2n+1-k)/k,$$
$$F_2^L(h) = f(h)\theta(-h-d);\ h_{2n}^L(\xi) = -\xi + 2nd/k - 2d;$$
$$F_3^L(h) = F_2^L(h);\ h_{3n}^L(\xi) = h_{2n}^L(\xi) + 2d/k;$$
$$F_4^L(h) = f(h)(\theta(-h) - \theta(-h-d));\ h_{4n}^L(\xi) = kh_{1n}^L(\xi);$$
$$F_5^L(h) = f(h)(\theta(h) - \theta(h+d));\ h_{5n}^L(\xi) = -h_{4n}^L(\xi);$$

Right of obstacle $x > 0$

$$\eta_R(x,t) = [f(\xi_R) + f(\nu_R)\theta(\nu_R)]/2 + \eta_R'(\nu_R)/[2(1+k)^2];$$
$$\eta_R'(\nu) = \sum_{n=0}^{\infty}[(1-k)/(1+k)]^{2n} \sum_{j=1}^{5} g_j F_j^R(h_{jn}^R(\nu));$$
$$F_1^R(h) = f(h)\theta(-h-d);\ h_{1n}^R(\nu) = \nu + d(2n+1-k)/k;$$
$$F_2^R(h) = f(h)\theta(h);\ h_{2n}^R(\nu) = -\nu - 2nd/k;$$
$$F_3^R(h) = F_2^R(h);\ h_{3n}^R(\nu) = h_{2n}^R(\nu) - 2d/k;$$
$$F_4^R(h) = f(h)\theta(h+d)\theta(-h);\ h_{4n}^R(\nu) = k\nu + 2nd;$$
$$F_5^R(h) = f(h)\theta(-h-d)\theta(h);\ h_{5n}^R(\nu) = -h_{4n}^R(\nu) - 2d,$$

Submerged Obstacle



Analytical expressions for wave amplitudes

Transmitted Waves $x < -d$

$$\eta_L(x,t) = -\left(\frac{1-k}{1+k}\right)^{2i-1} \frac{2k}{(k+1)^2} f(t - x - 2id/k), \quad t > x_0 + 2id/k, \; i = 1,2,3....$$

Above obstacle $-d < x < 0$, positive amplitude

$$\eta_M(x,t) = \left(\frac{1-k}{1+k}\right)^{2i} \frac{1}{(k+1)} f(t + x/k - 2id/k), \; i = 0,1,2.......$$

Above obstacle $-d < x < 0$, negative amplitude

$$\eta_M(x,t) = -\left(\frac{1-k}{1+k}\right)^{2i-1} \frac{1}{(k+1)} f(t - x/k - 2id/k), \; i = 1,2,3.....$$

Reflected Waves $x > 0$

$$\eta_R(x,t) = \left(\frac{1-k}{1+k}\right)^{2i} \frac{2k}{(k+1)^2} f(t + x + d - (2i+1)d/k), \quad t > x_0 + (2i+1)d/k, \; i = 0,1,2,3....$$

Wave Resonance

Two general waves above shelf

$$\eta_M(x,t) = \left(\frac{1-k}{1+k}\right)^2 \frac{1}{(k+1)} f_1(t + x/k - 2d/k) + \frac{1}{1+k} f_2(t + x/k)$$

Spatial Condition for Resonance

$$x_1 = x_0 + 2d/k$$

After superposition

$$\eta_M(x,t) = \frac{2(1+k^2)}{(k+1)^3} f(t + x/k - 2d/k)$$

Transmitted Waveform

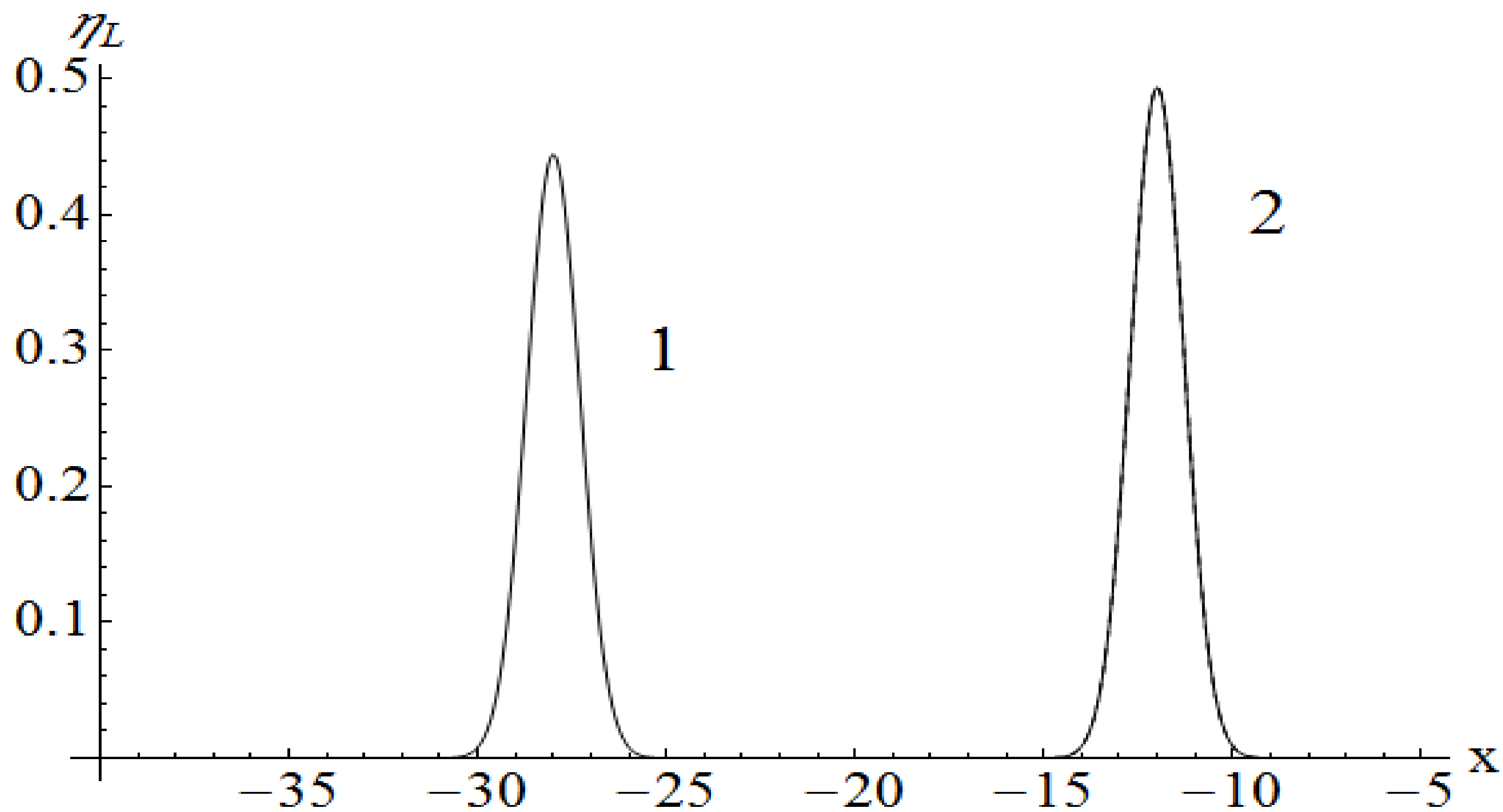
$$\eta_L(x,t) = \frac{4k(1+k^2)}{(k+1)^4} \cdot f(t + x + d - 3d/k)$$

Transmission Coefficient

$$T = \frac{2(1+k^2)}{(k+1)^3} \cdot \frac{2k}{(1+k)}$$

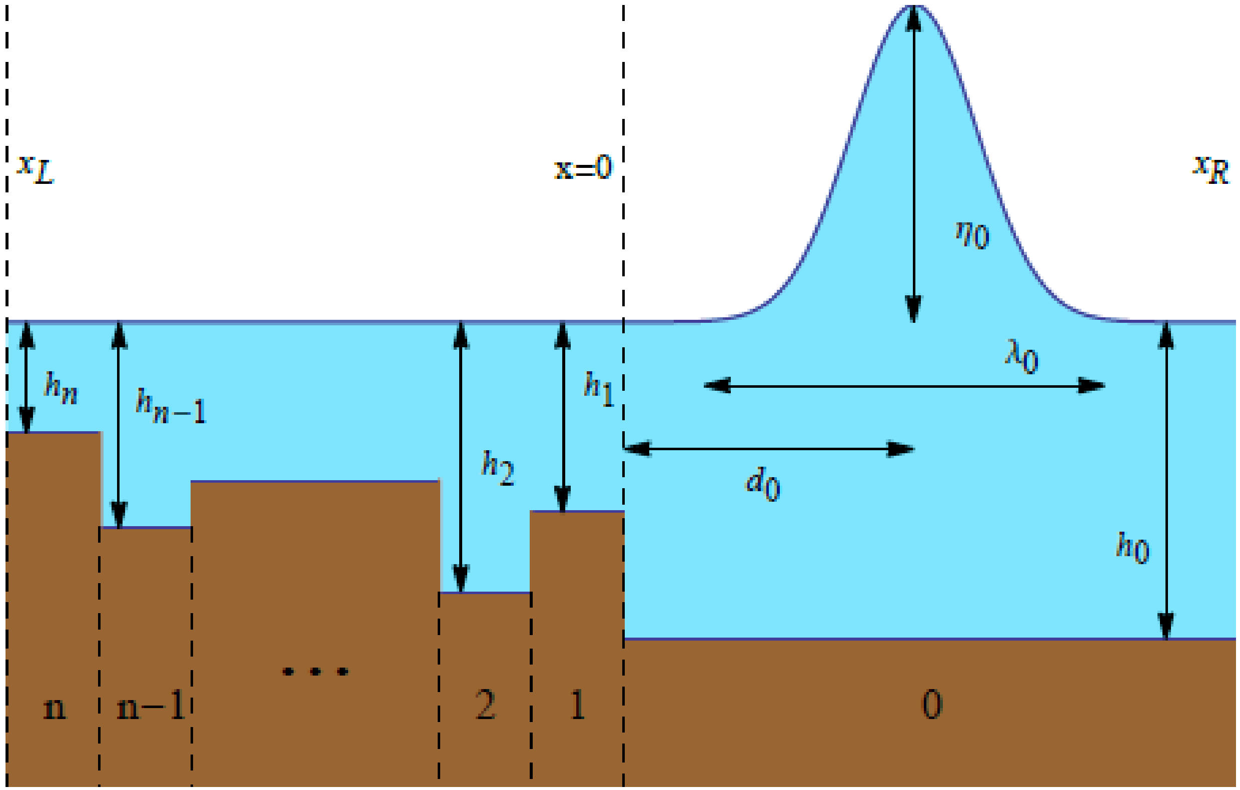
Wave Resonance

First two transmitted waves



Numerical solution for arbitrary piecewise seafloor

$$\varphi(x) = \begin{cases} k_0^2 : x > 0 \\ k_1^2 : x_1 < x < 0 \\ \cdots \\ k_n^2 : x < x_{n-1}. \end{cases}$$



Numerical solution for arbitrary piecewise seafloor

Finite Difference Method (second-order Lax-Wendroff scheme)

$$\vec{V}_j^{n+1} = \vec{V}_j^n - \frac{\Delta t}{2\Delta x} A(\vec{V}_{j+1}^n - \vec{V}_{j-1}^n) + \frac{\Delta t^2}{2\Delta x^2} A^2(\vec{V}_j^n - 2\vec{V}_{j-1}^n + \vec{V}_{j-2}^n)$$

$$\vec{V} = \begin{pmatrix} \eta \\ u \end{pmatrix}$$

Boundary Conditions at seafloor steps

$$A = \begin{pmatrix} 0 & \varphi(x) \\ 1 & 0 \end{pmatrix}$$

$$\lim_{\delta x \to 0^+} \eta(\delta x, t) = \lim_{\delta x \to 0^+} \eta(-\delta x, t)$$

$$\lim_{\delta x \to 0^+} \varphi(\delta x) u(\delta x, t) = \lim_{\delta x \to 0^+} \varphi(-\delta x) u(-\delta x, t)$$

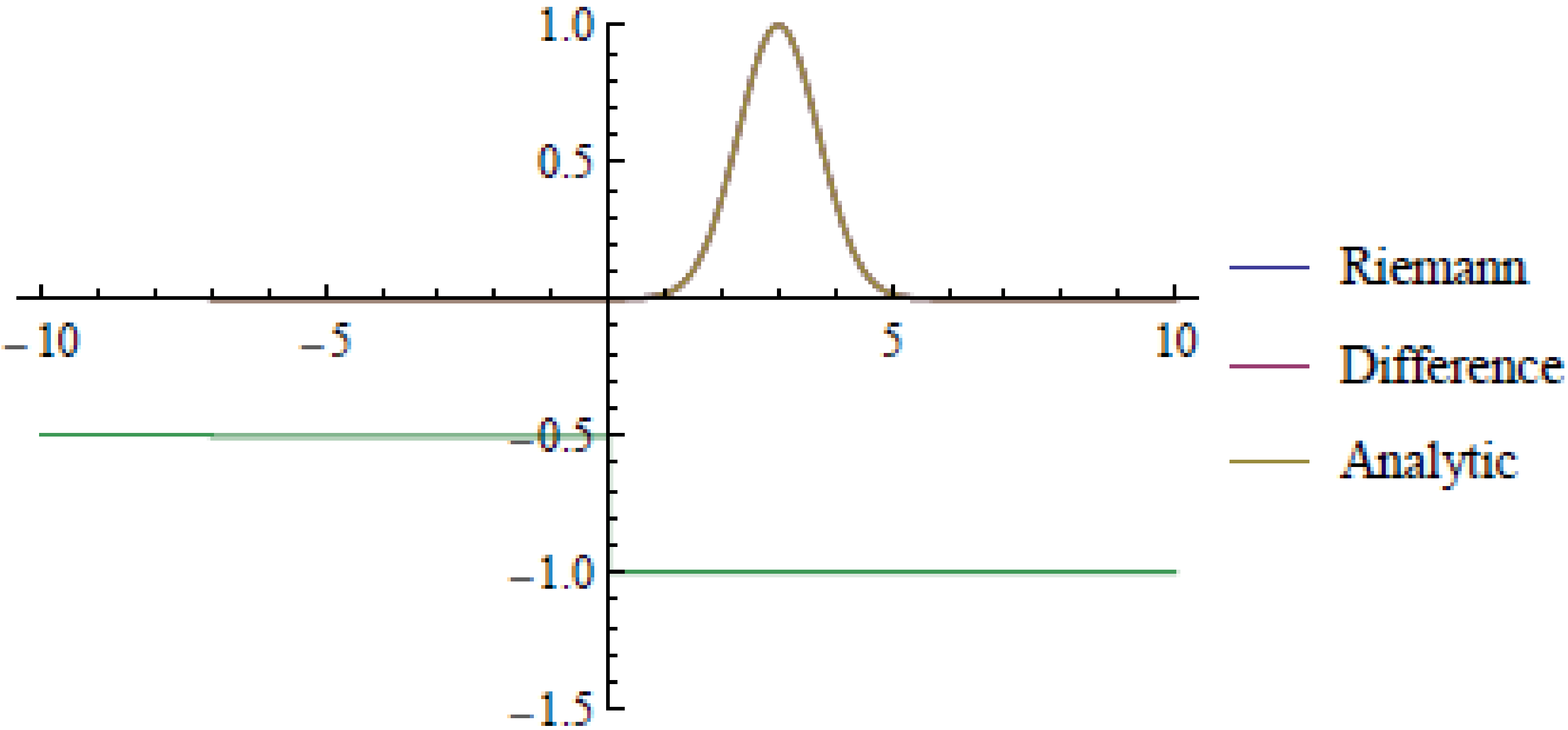
$$\varphi(\delta x) \frac{\partial \eta}{\partial x}(\delta x, t) = \varphi(-\delta x) \frac{\partial \eta}{\partial x}(-\delta x, t)$$

$$\lim_{\delta x \to 0^+} \eta(\delta x, t) = \lim_{\delta x \to 0^+} \eta(-\delta x, t)$$

Radiation conditions at computational boundaries (Beam-Warming scheme)

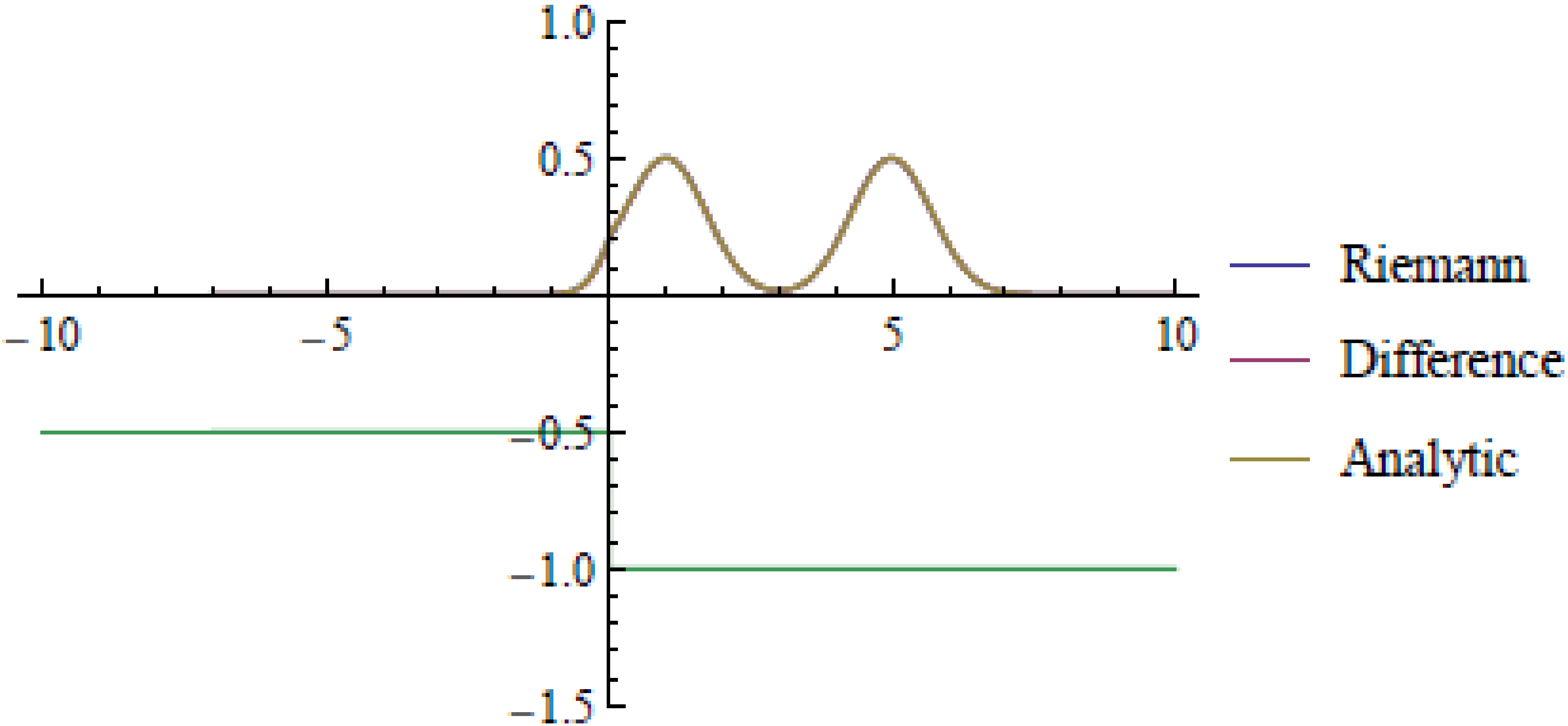
$$\vec{V}_{\mathbf{1}}{}_j^{n+1} = \vec{V}_{\mathbf{1}}{}_j^n - \frac{\Delta t}{2\Delta x_1} A(3\vec{V}_{\mathbf{1}}{}_j^n - 4\vec{V}_{\mathbf{1}}{}_{j-1}^n + \vec{V}_{\mathbf{1}}{}_{j-2}^n) + \frac{\Delta t^2}{2\Delta x_1^2} A^2(\vec{V}_{\mathbf{1}}{}_j^n - 2\vec{V}_{\mathbf{1}}{}_{j-1}^n + \vec{V}_{\mathbf{1}}{}_{j-2}^n)$$

Comparison to Analytical Solution



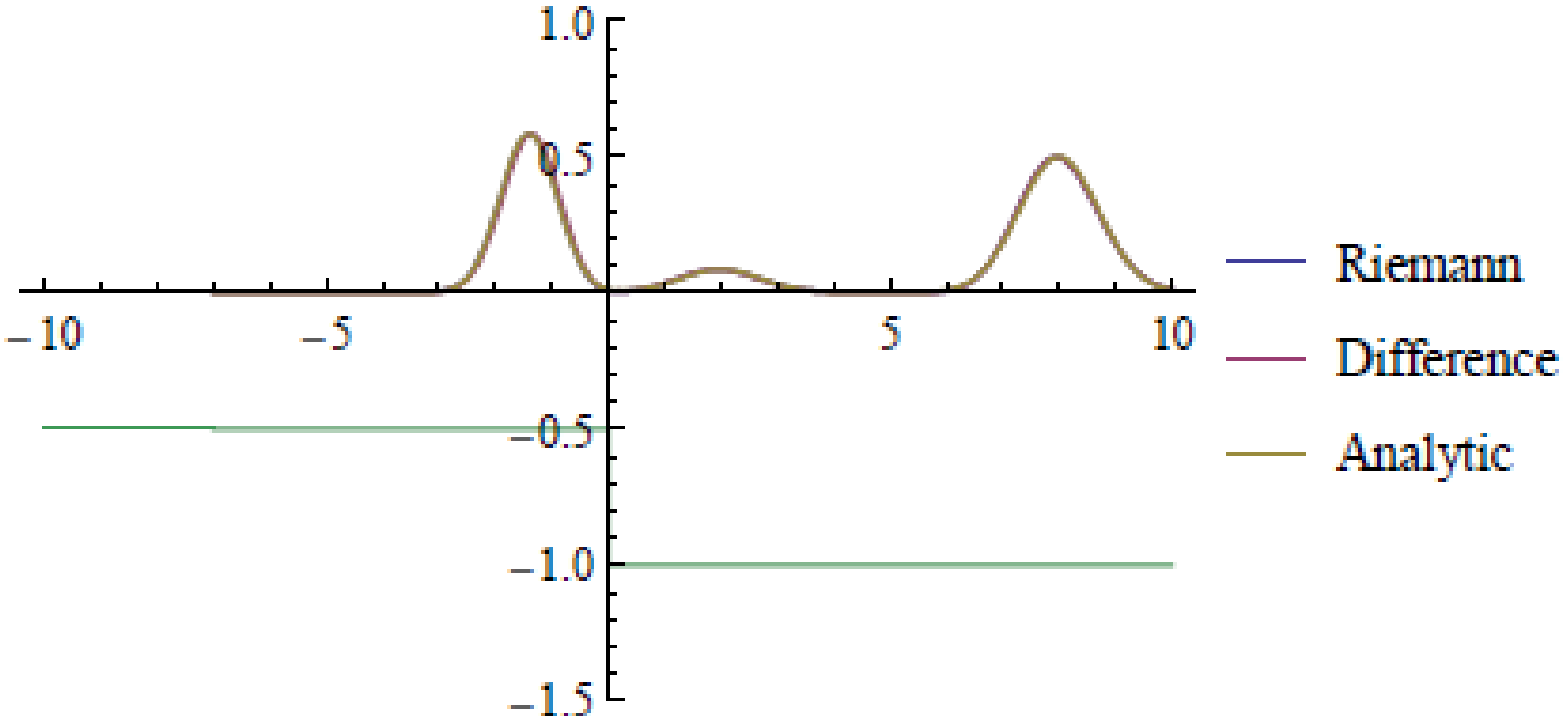
(a) $t = 0$

Comparison to Analytical Solution



(b) $t = 2$

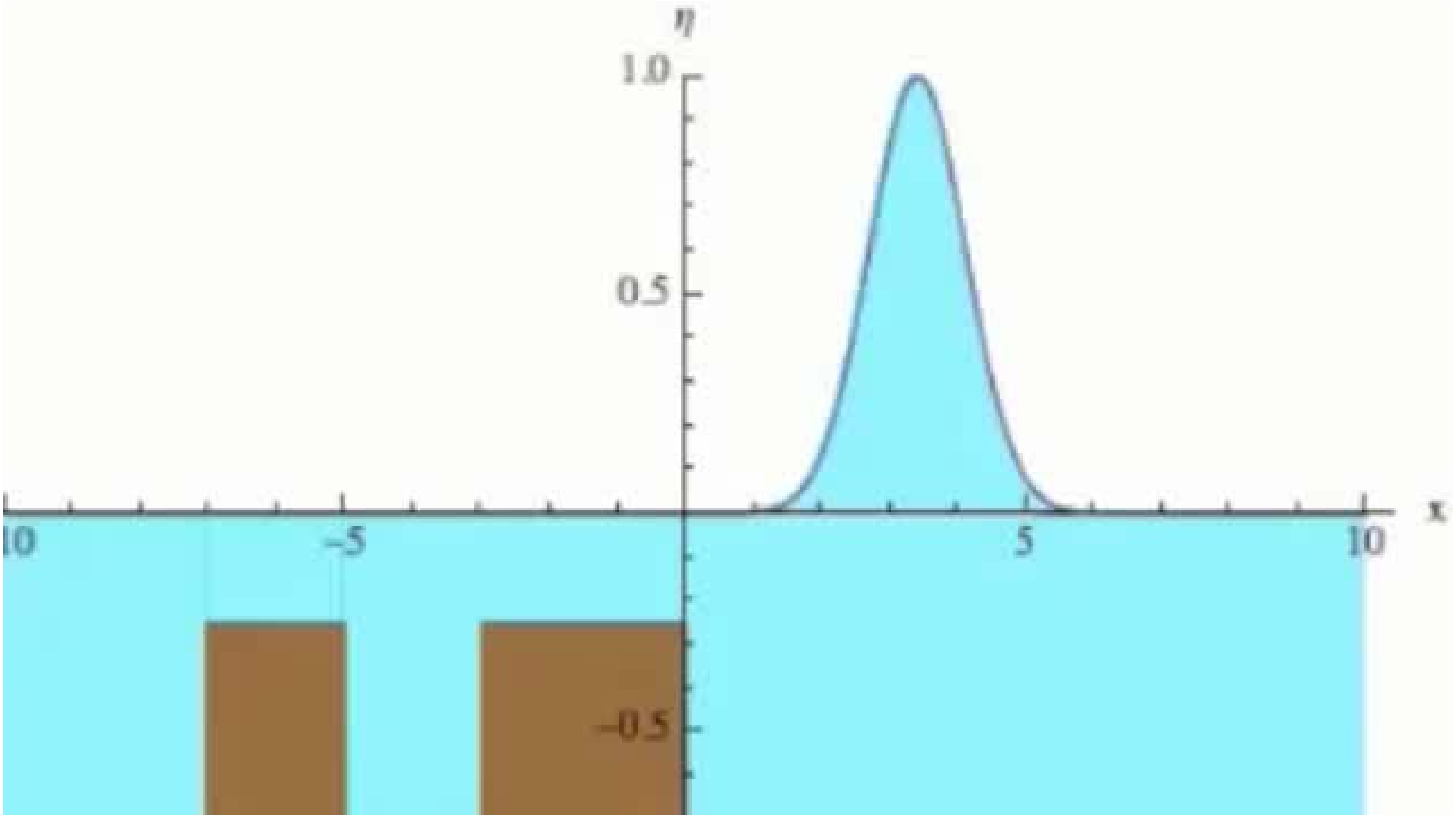
Comparison to Analytical Solution



(c) $t = 5$

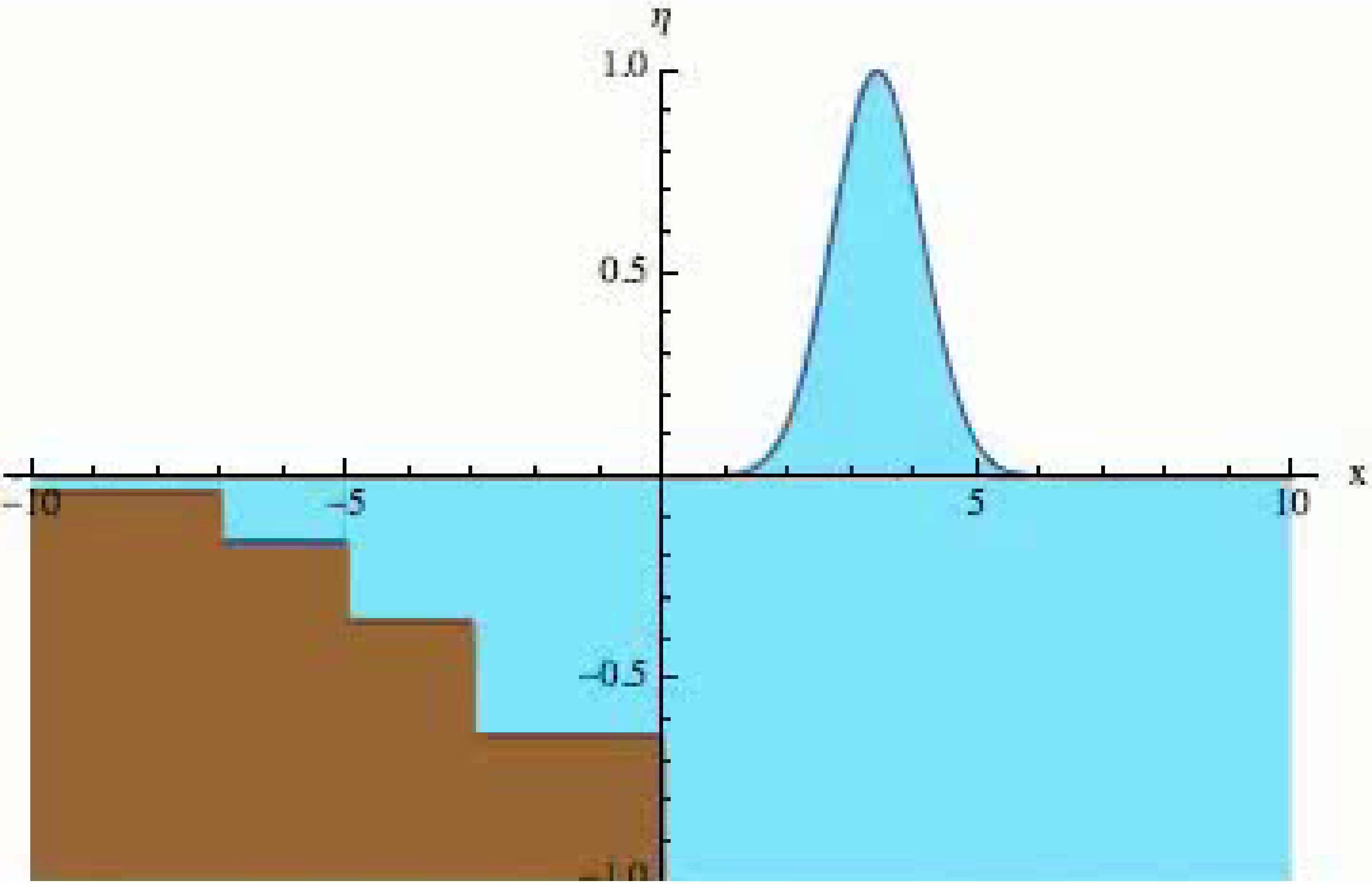
More General Seafloor Topographies

Two Submerged Obstacles



More General Seafloor Topographies

beach



Results and Conclusions

- Derivation of momentum conjugation condition at step discontinuity.
- Exact linear wave solutions for shelf and submerged obstacle.
- Explicit solutions for nonlinear perturbations with renormalized characteristics. Favorable comparison with exact Riemann wave solutions.
- Analytical expressions for wave amplitudes above the shelf and transmitted waves
- Criterion for resonance and transmission coefficient
- Ability to model wave propagation over more realistic seafloor topographies