Introduction to boundary integral equation methods

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BIERS Meeting
September 25, 2018
What are boundary integral equations?

• We can reformulate boundary value problems for PDEs in a domain as integral equations on the boundary of that domain.

• We typically use them for linear, elliptic, and homogeneous PDEs, but not always.

• Boundary integral equation methods refer to the numerical solution of these integral equations.
Why study boundary integral equations?

• Since we only solve a problem on the boundary, there is an overall reduction in dimension by one.

• It is “meshless” so it can compute solutions in complicated domains.

• Numerics rule of thumb: “It’s better to integrate than to differentiate.”
What are the challenges?

• The boundary integral formulation is initially more abstract/less intuitive.

• Numerical solution of PDEs yield *sparse matrices*, while numerical solutions of boundary integral equations yield *dense matrices*.

• Technical challenges regarding error analysis.
Trying to build some intuition

How can we reduce the solution of a PDE in a domain to an BIE on its boundary?

Consider the two-point boundary value problem in one dimension.

\[ u''(x) = 0, \quad \text{in } (0, 1), \]

\[ u = \alpha, \quad \text{on } x = 0, \]

\[ u = \beta, \quad \text{on } x = 1. \]

The solution we seek must satisfy (1) the DE and (2) the BCs.
Trying to build some intuition

We write the solution as a linear combination of two functions that satisfy the DE.

\[ u(x) = c_1 u_1(x) + c_2 u_2(x), \quad x \in (0, 1), \]
\[ u_1(x) = |x|, \quad u_2(x) = |1 - x|. \]

By requiring that this function satisfies the BCs, we find that the coefficients must satisfy

\[ u(0) = \alpha = c_1 u_1(0) + c_2 u_2(0) \quad \implies \quad c_2 = \alpha \]
\[ u(1) = \beta = c_1 u_1(1) + c_2 u_2(1) \quad \implies \quad c_1 = \beta. \]
Trying to build some intuition

• For a boundary value problem, we need to satisfy the differential equation and the boundary conditions.

• Suppose we seek the solution as a superposition of solutions of the differential equation. Then that solution automatically satisfies the differential equation.

• All that is left is to satisfy the boundary conditions.
Extending this idea

Suppose we would like to solve the interior, Dirichlet boundary value problem in two/three dimensions.

\[ \Delta u = 0, \quad \text{in } D, \]
\[ u = f \quad \text{on } B. \]

Can we write the solution as a “superposition” of functions that satisfy Laplace’s equation, and then require that this superposition satisfies the boundary conditions?
We write the solution as the *continuous* superposition,

\[ u(x) = \frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n} \mu(y) \mathrm{d}\sigma_y, \quad x \in D, \quad n = 2, 3. \]

It can be shown that

\[ \Delta \left[ \frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n} \mu(y) \mathrm{d}\sigma_y \right] = 0, \quad x \in D. \]
Boundary integral equation

Now that we have a function that satisfies the PDE, we require it to satisfy the BC. However, we note that

$$\lim_{x \to y^*} \frac{1}{2^{n-1} \pi} \int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n} \mu(y) \, d\sigma_y = u(y^*) - \frac{1}{2} \mu(y^*), \quad y^* \in B.$$ 

It follows that the density satisfies

$$\frac{1}{2} \mu(y^*) + \frac{1}{2^{n-1} \pi} \int_B \frac{\nu_y \cdot (y^* - y)}{|y^* - y|^n} \mu(y) \, d\sigma_y = f(y^*), \quad y^* \in B.$$
Boundary integral equation method

1. Solve boundary integral equation,

\[ \frac{1}{2} \mu(y^*) + \frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (y^* - y)}{|y^* - y|^n} \mu(y) \, d\sigma_y = f(y^*), \quad y^* \in B. \]

2. Evaluate the solution,

\[ u(x) = \frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n} \mu(y) \, d\sigma_y, \quad x \in D. \]
What have we skipped?

• We have introduced a solution in terms of the *fundamental solution* of the PDE (Green’s functions, Green’s theorems, etc.).

• This solution has a specific *jump* when evaluated on the boundary (Gauss’ theorem).

• How to compute the numerical solution of the boundary integral equation and how to compute the numerical evaluation of the solution?

• *Applications where this method is useful.*
What are my interests in BIE methods?

With BIE methods, we have an explicit expression for the solution, e.g.

\[ u(x) = \frac{1}{2^{n-1} \pi} \int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n} \mu(y) d\sigma_y, \quad x \in D. \]

That means that this representation contains all of the physical behavior of a system.

Can we use this to extract valuable insight into complex problems through asymptotic analysis of this expression?
Any Questions?