Introduction to boundary integral equation methods

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What are boundary integral equations?

- We can reformulate boundary value problems for *PDEs in a domain* as *integral equations on the boundary* of that domain.

- We typically use them for *linear, elliptic, and homogeneous PDEs*, but not always.

- *Boundary integral equation methods* refer to the numerical solution of these integral equations.

Why study boundary integral equations?

- Since we only solve a problem on the boundary, there is an overall reduction in dimension by one.

- It is “meshless” so it can compute solutions in complicated domains.

- Numerics rule of thumb: “It’s better to integrate than to differentiate.”

What are the challenges?

- The boundary integral formulation is initially more abstract/less intuitive.

- Numerical solution of PDEs yield *sparse matrices*, while numerical solutions of boundary integral equations yield *dense matrices*.

- Technical challenges regarding error analysis.

Trying to build some intuition

How can we reduce the solution of a PDE in a domain to an BIE on its boundary?

Consider the two-point boundary value problem in one dimension.

$$
\begin{aligned}
u''(x) &= 0, \quad \text{in } (0,1), \\
u &= \alpha, \quad \text{on } x = 0, \\
u &= \beta, \quad \text{on } x = 1.
\end{aligned}
$$

The solution we seek must satisfy (1) the DE and (2) the BCs.

Trying to build some intuition

We write the solution as a linear combination of two functions that satisfy the DE.

$$u(x) = c_1 u_1(x) + c_2 u_2(x), \quad x \in (0,1),$$
$$u_1(x) = |x|, \quad u_2(x) = |1-x|.$$

By requiring that this function satisfies the BCs, we find that the coefficients must satisfy

$$u(0) = \alpha = c_1 u_1(0) + c_2 u_2(0) \quad \Longrightarrow \quad c_2 = \alpha$$
$$u(1) = \beta = c_1 u_1(1) + c_2 u_2(1) \quad \Longrightarrow \quad c_1 = \beta.$$

Trying to build some intuition

- For a boundary value problem, we need to satisfy the differential equation and the boundary conditions.

- Suppose we seek the solution as a superposition of solutions of the differential equation. Then that solution automatically satisfies the differential equation.

- All that is left is to satisfy the boundary conditions.

Extending this idea

Suppose we would like to solve the interior, Dirichlet boundary value problem in two/three dimensions.

$$\begin{aligned} \Delta u &= 0, \quad \text{in } D, \\ u &= f \quad \text{on } B. \end{aligned}$$

Can we write the solution as a “superposition” of functions that satisfy Laplace’s equation, and then require that this superposition satisfies the boundary conditions?

Skipping some analysis…

We write the solution as the *continuous* superposition,

$$u(x) = \frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (x-y)}{|x-y|^n} \mu(y) \mathrm{d}\sigma_y, \quad x \in D, \quad n = 2, 3.$$

kernel: $\frac{\nu_y \cdot (x-y)}{|x-y|^n}$

density: $\mu(y)$

It can be shown that

$$\Delta \left[\frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (x-y)}{|x-y|^n} \mu(y) \mathrm{d}\sigma_y \right] = 0, \quad x \in D.$$

Boundary integral equation

Now that we have a function that satisfies the PDE, we require it to satisfy the BC. However, we note that

$$\lim_{\substack{x\to y^\star \\ x\in D}} \frac{1}{2^{n-1}\pi}\int_B \frac{\nu_y\cdot(x-y)}{|x-y|^n}\mu(y)\mathrm{d}\sigma_y = u(y^\star) - \frac{1}{2}\mu(y^\star), \quad y^\star\in B.$$

It follows that the density satisfies

$$\frac{1}{2}\mu(y^\star) + \frac{1}{2^{n-1}\pi}\int_B \frac{\nu_y\cdot(y^\star-y)}{|y^\star-y|^n}\mu(y)\mathrm{d}\sigma_y = f(y^\star), \quad y^\star\in B.$$

Boundary integral equation method

1. Solve boundary integral equation,

$$\frac{1}{2}\mu(y^\star) + \frac{1}{2^{n-1}\pi}\int_B \frac{\nu_y \cdot (y^\star - y)}{|y^\star - y|^n}\mu(y)\mathrm{d}\sigma_y = f(y^\star), \quad y^\star \in B.$$

2. Evaluate the solution,

$$u(x) = \frac{1}{2^{n-1}\pi}\int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n}\mu(y)\mathrm{d}\sigma_y, \quad x \in D.$$

What have we skipped?

- We have introduced a solution in terms of the *fundamental solution* of the PDE (Green’s functions, Green’s theorems, etc.).
- This solution has a specific *jump* when evaluated on the boundary (Gauss’ theorem).
- How to compute the numerical solution of the boundary integral equation and how to compute the numerical evaluation of the solution?
- *Applications where this method is useful.*

What are my interests in BIE methods?

With BIE methods, we have an explicit expression for the solution, *e.g.*

$$u(x) = \frac{1}{2^{n-1}\pi} \int_B \frac{\nu_y \cdot (x - y)}{|x - y|^n} \mu(y) \mathrm{d}\sigma_y, \quad x \in D.$$

That means that this representation contains *all* of the physical behavior of a system.

Can we use this to extract valuable insight into complex problems through asymptotic analysis of this expression?

Any Questions?