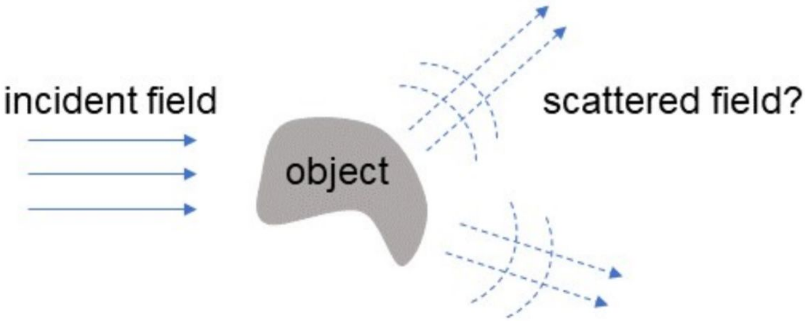
Time Dependent Scattering by a Sound-Hard Sphere

Tomas U. Virgen
Department of Applied Mathematics, University of California, Merced
Supported by CAMP
November 12, 2020

Direct Scattering Problem

- We want to solve the scattering problem for a “sound-hard” sphere.



- We consider a stationary sphere at first, but we want to use a method that can easily extend to a moving scatterer.

Governing Equation

- Suppose we have a sphere of radius a centered at r_c. Mathematically, we solve this scattering problem by solving the following initial-boundary value problem for the wave equation:

$$u_{tt} - c^2 \Delta u = 0 \quad \text{in } |\mathbf{r} - \mathbf{r}_c| > a \text{ and } t > 0,$$

$$u(\mathbf{r}, 0) = f(\mathbf{r}) \ \ \text{on } |\mathbf{r} - \mathbf{r}_c| > a,$$

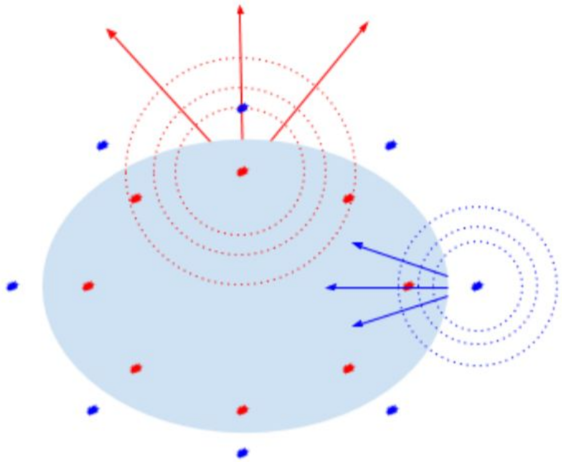
$$u_t(\mathbf{r}, 0) = g(\mathbf{r}) \ \ \text{on } |\mathbf{r} - \mathbf{r}_c| > a,$$

$$\partial_n u = 0 \quad \text{on } |\mathbf{r} - \mathbf{r}_c| = a \text{ and } t > 0.$$

← Sound-hard boundary condition

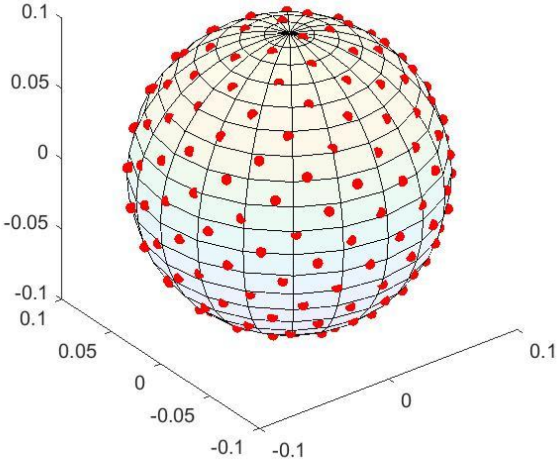
Method of Fundamental Solutions

- We can apply the method of fundamental solutions to this problem, whereby we approximate the scattered field by a superposition of finitely many fundamental solutions, each of which exactly solves the wave equation.



Mathon, Rudolf and Johnston, R.L. “The Approximate Solution of Elliptic Boundary-Value Problems by Fundamental Solutions.” SIAM Journal on Numerical Analysis, vol. 14, no. 4, 1977, pp. 638-650

Constructing our Scattering Sphere



- The basic idea is to place discrete sources, each occupying a finite volume, inside and near the boundary of the sphere. They are placed *inside* the sphere so that they are *outside* of the domain.
- To be concrete, let's suppose we have P discrete sources(which we can denote by Q_j) within the sphere, each of which is centered at $r_j^c, \quad j = 1, 2, \ldots, P$
- We found that the most optimal way to place points around the sphere was by applying a Fibonacci lattice algorithm.

González, Álvaro. "Measurement of Areas on a Sphere Using Fibonacci and Latitude–Longitude Lattices." Mathematical Geosciences, vol. 42, no. 1, 2009, pp. 49–64.

Constructing Our Fundamental Solutions

- Suppose that each discrete sources is of the form

$$Q(\mathbf{r},t) = \begin{cases} C & |\mathbf{r}| < a, t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases}$$

- Such a source placed within the sphere would satisfy the inhomogeneous wave equation

$$v_{tt} - c^2 \Delta v = Q(\mathbf{r},t) \quad |r - r_i^c| < a, t \in [t_1, t_2]$$

$$v(\mathbf{r},0) = 0, \quad v_t(\mathbf{r},0) = 0$$

Constructing Our Fundamental Solutions

- We introduce the Green's function for the three-dimensional wave equation

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \frac{1}{4\pi c|\mathbf{r} - \mathbf{r}_0|}\delta\left(|\mathbf{r} - \mathbf{r}_0| - c(t - t_0)\right)$$

- The solution to our inhomogeneous wave equation can then be given in terms of this Green's function:

$$v(\mathbf{r}, t) = \int_0^t \iiint\limits_{\mathbb{R}^3} G(\mathbf{r}, t; \mathbf{r}_0, t_0) Q(\mathbf{r}_0, t_0) \mathrm{d}\mathbf{r}_0 \mathrm{d}t_0$$

- It is left as an exercise to show that this integral gives way to

$$v(\mathbf{r}, t) = \begin{cases} \frac{1}{4\pi c^2|\mathbf{r}|} & \text{if } t \text{ lies in the interval } [t_1 + \frac{|\mathbf{r}|}{c}, t_2 + \frac{|\mathbf{r}|}{c}] \\ 0 & \text{otherwise} \end{cases}$$

Full Scattered Field

- The full scattered field can then be written as a linear combination of spherical waves centered at each discrete source with unknown coefficients:

$$u^s(r, t_k) = \sum_{i=1}^{k} \sum_{j=1}^{P} c_{ij} v_j(r - r_j^c, t)$$

$\sum_{i=1}^{k} \leftarrow$ Sum over each discrete time interval up to k

$\sum_{j=1}^{P} \leftarrow$ Sum over each discrete source

- The unknown coefficients can be found by applying the “sound-hard” boundary condition to each source for each discrete time interval.

Sound-Hard Boundary Condition

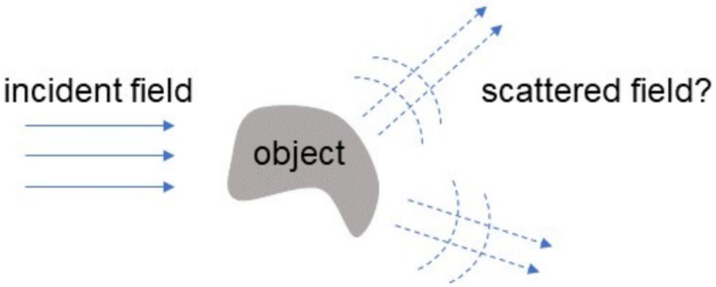
- Recall that the sound-hard boundary condition is given by

$$\partial_n u = 0 \ \text{ on} |\mathbf{r} - \mathbf{r}_c| = a \text{ and } t > 0$$

- The solution to the full scattering problem actually takes the sum of both the incident and scattered field

$$u = u^s + u^{inc}$$

$$u^{inc} = \cos(z - ct)$$



Sound-Hard Boundary Condition

- We can reformulate the sound-hard boundary condition in terms of the incident and scattered fields

$$\partial_n u = \partial_n u^s + \partial_n u^{inc} = 0$$
$$\rightarrow \partial_n u^s = -\partial_n u^{inc}$$

- The normal derivative of the incident field is "trivial" given its simple form. As for the scattered field,

$$\partial_n u^s \approx \sum_{i=1}^{k} \sum_{j=1}^{P} c_{ij} \partial_n v(r - r_j^c, t)$$

Linear System

- Suppose we have P discrete sources that are piecewise continuous over N discrete time intervals. If we evaluate the normal derivative for *each* source at *each* discrete time interval, we actually get a system of linear equations that we can write in a matrix form

$$\boldsymbol{U}^s\boldsymbol{C} = -\boldsymbol{U}^{inc}$$

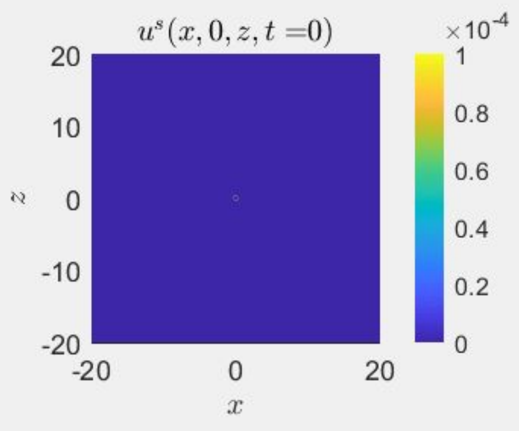
$$\boldsymbol{C} = -(\boldsymbol{U}^s)^{-1}\boldsymbol{U}^{inc}$$

$$\boldsymbol{U}^s_{ij} = \partial_n v(r^b_j - r^c_i, t) = \begin{bmatrix} \partial_n v(r^b_1 - r^c_1, t) & \dots & \partial_n v(r^b_1 - r^c_P, t) \\ \vdots & \ddots & \vdots \\ \partial_n v(r^b_P - r^c_1, t) & \dots & \partial_n v(r^b_P - r^c_P, t) \end{bmatrix}$$

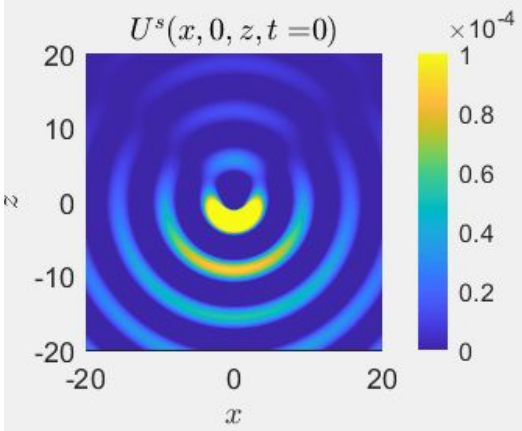
$$\boldsymbol{C} = C_{ij} = \begin{bmatrix} C_{11} & \dots & C_{1N} \\ \vdots & \ddots & \vdots \\ C_{P1} & \dots & C_{PN} \end{bmatrix}$$

$$\boldsymbol{U}^{inc}_{ij} = \partial_n u^{inc}(r^b_i, t_j) = \begin{bmatrix} \partial_n u^{inc}(r^b_1, t_1) & \dots & \partial_n u^{inc}(r^b_1, t_N) \\ \vdots & \ddots & \vdots \\ \partial_n u^{inc}(r^b_P, t_1) & \dots & \partial_n u^{inc}(r^b_P, t_N) \end{bmatrix}.$$

Stationary Scattering Sphere Results



xz-plane colormap GIF of approximated scattered field. a=0.1, P=401.

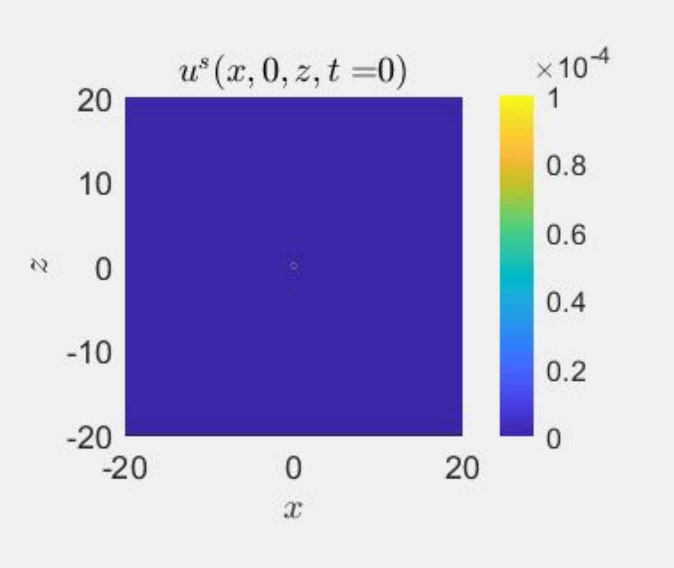


xz-plane colormap GIF of analytic scattered field.

- When we plot the analytic solution and our approximation side by side, they look quite visually similar

Moving Scatterer Results

- It is relatively straightforward to extend the results to a moving scatterer. In the case of a sphere moving along the positive z-axis:



Conclusion

- We find that our approximation of the scattered field is visually similar to the analytic expression for the field.
- Overall, our results demonstrate the usefulness of the application of the method of fundamental solutions to scattering problems.
- Although our method is not novel, it can be shown to apply to more novel cases.
- Further work must be done in order to precisely quantify the error between the two fields.
- Moreover, the application of this approximation to different scattering problems must also be considered for future studies. It would be important to study how changes in the parameters, such as the number of sources and their arrangement, would correspond to different scattering scenarios.

Acknowledgements

- Done in collaboration with Dr. Arnold Kim as part of the California Alliance for Minority Participation(CAMP)

References

1. González, Álvaro. “Measurement of Areas on a Sphere Using Fibonacci and Latitude–Longitude Lattices.” Mathematical Geosciences, vol. 42, no. 1, 2009, pp. 49–64.
2. Mathon, Rudolf and Johnston, R.L. “The Approximate Solution of Elliptic Boundary-Value Problems by Fundamental Solutions.” SIAM Journal on Numerical Analysis, vol. 14, no. 4, 1977, pp. 638-650