Preliminary Exam in Ordinary Differential Equations, UC Merced, January 2016

Directions: You have 4 hours to complete this exam. Show all of your work on separate sheets of papers, and circle your final answers when appropriate. You are allowed one hand-written crib sheet, but NO calculators, phones, or other study aides are allowed. Remember to explain your work clearly and legibly so that you may receive full credit.

1. Consider the following equation for $x(t)$:

$$\frac{dx}{dt} = x^\alpha$$

   with $x(0) \geq 0$ and $\alpha > 0$. Show that the only value of α for which the equation has solutions that are both unique and exist for all time is $\alpha = 1$.

2. Show that any equation that can be written in the form

$$f(x) + g(y)\frac{dy}{dx} = 0$$

   is exact, and find its solution $y(x)$ in terms of integrals of f and g.

3. Find one power series solution of the equation

$$x(1-x)y'' - 3xy' - y = 0.$$

   By summing the power series solution you find, you should be able to express the solution in closed form. Using this closed form solution, find a second solution.

4. Assuming $\lambda > 0$ and initial conditions $\{x_1(0) = a, x_2(0) = b\}$, find the solution of the following coupled system:

$$\frac{dx_1}{dt} = x_1 + \lambda x_2$$
$$\frac{dx_2}{dt} = x_2$$

   Using this general solution, compute the matrix exponential

$$\exp\left(\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}\right).$$

5. Find a particular solution of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 12e^{-t} - 6e^t.$$

6. For all positive values of c find all the fixed points of

$$\frac{dx}{dt} = \sin x + c,$$

   and determine analytically which are stable and unstable. Draw the part of the phase diagram between $-\pi$ and π. There are three different cases, $0 \leq c < 1$, $c = 1$, and $c > 1$. Be careful with the $c = 1$ case.

7. Draw the phase portrait for the nonlinear system

$$\frac{dx}{dt} = x(1-y)$$
$$\frac{dy}{dt} = y(2x-4)$$

and find the equation that x and y satisfy along each trajectory. (This should be an equation of the form $f(x,y) = C$ where C is a constant; each trajectory should then correspond to the set of points that satisfy the equation for a fixed value of C.)

8. Consider the following nonlinear system:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\sin x - ky|y|.$$

Let
$$E = \frac{1}{2}y^2 - \cos x.$$

Show that
$$\frac{dE}{dt} = -ky^2|y|.$$

What are the consequences of this fact for the dynamics of the nonlinear system near $(0,0)$?