Calculus Preliminary Exam *January 2022*

Duration: 240 minutes
Instructions: Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. Determine as accurately as possible the limit as $n \to \infty$ of sequences with following general terms:

   (a) $a_n = \frac{\cos(\pi n)}{n}$

   (b) $b_{n+1} = b_n - \alpha\, b_{n-1}$, with $b_0 = 0$, $b_1 = 1$ as a function of α.
   Note that one may write $b_n = c_1 r_1^n + c_2 r_2^n$ for appropriately chosen constants c_1, c_2, r_1, and r_2.

   (c)
   $$c_n = \sum_{k=0}^{k=n} (-1)^{k+1}(k+1)x^k$$
   for $|x| < 1$.

2. Compute the following limits:

   (a)
   $$\lim_{x \to 0} \left(\frac{\cos x - 1}{x^2} \right) \left(\frac{2 + 2x}{e^{2x}} \right)$$

   (b)
   $$\lim_{x \to 0} \frac{\sin x}{|x|}$$

   (c)
   $$\lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2}$$

3. Consider functions $f(x)$ that are monotonically increasing and such that $\lim_{x \to \infty} f(x) = \infty$. Determine what possible values the limit below can take and give an example of a corresponding function $f(x)$ yielding that limit
   $$\lim_{x \to \infty} \frac{f(x)}{\int_0^x f(t)dt}.$$

4. Calculate the following derivatives. Recall that $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$.

   (a)
   $$\frac{d}{dx} \left(\int_0^1 \frac{2}{\sqrt{\pi}} e^{-(x+t)^2} dx \right)$$

   (b)
   $$\frac{d}{dt} \left(\int_0^1 \frac{2}{\sqrt{\pi}} e^{-(x+t)^2} dx \right)$$

   (c)
   $$\frac{d}{dt} \left(\int_0^t \frac{2}{\sqrt{\pi}} e^{-(x+t)^2} dx \right)$$

   (d) $\frac{dy}{dx}$ given that $x^2 + (x+y)^2 = 1$.

5. Compute the following integrals
   (a) By parts: $\int e^{2x}\cos(3x)dx$
   (b) Anyway you like: $\int_0^1 \frac{x\,dx}{(1-x^2)^{3/2}}$
6. Surfaces and tangent planes
   (a) Sketch the surface given by $z = f(x,y) = x^4 - 2y^4$.
   (b) Give an expression for the tangent plane to the surface $f(x,y) = z + z^3$ at the point $(0,1,-1)$.
7. Show that the volume V of a straight cone (with circular cross-section and symmetry axis perpendicular to the cross-section) of radius R and height h is given by $V = \pi R^2 h$.
8. Consider that x-axis is the West-East axis and that the y-axis is the South-North axis. The density of pollution collected by a single passage of a filter fitted to an airplane is given by $d(x,y,z) = 5 + x^2 + y^2 - z^2 + xy^2$ (g/km). How much pollution would be collected by a plane traveling at a height of $z = 1$ km and tracing the northern half of a semi-circle of radius 3 km centered at the origin?
9. Consider a closed curve C made of $y = 0$ for $-1 \leq x \leq 1$ and the upper half-circle $x^2 + y^2 = 1$, traced counter-clockwise and a vector field $\vec{F}(x,y)$
   (a) Explain how you would compute the work $\oint_C \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$ for a conservative vector field without parametrizing C.
   (b) Explain how you would compute the work $\oint_C \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$ for a non-conservative vector field without parametrizing C.
   (c) Let $\vec{F} =< xy + y^2, 3x - y^2 >$. Determine if the field $\vec{F}$ is conservative or not and compute the work.
10. Consider the part of the surface $z = 4 - x^2 - y^2$ (in meters) oriented with its normal pointing up that lies above the unit circle centered at the origin in the xy–plane. What is the flux of water into that surface if the water velocity is $\vec{F}(x,y,z) = (2x, 2y, -4z+1)$m/s? How long would it take to get 31.416 m^3 of water through this surface?
11. General questions
   (a) The vector field $\vec{F} =< y^2 e^x + \sin x,\ 2ye^x + 3y^2 >$ is conservative. Find its potential.
   (b) State the divergence (or Gauss's) theorem when applied to a one-dimensional domain which is a segment of the x-axis.
   (c) Give a non-constant two-dimensional vector field $\vec{F}$ whose magnitude is always 3.
   (d) Give two properties of the gradient of a function.
   (e) Give the formula of a continuous function of one variable with a local maximum, a horizontal asymptote at $y = 2$ and another one at $y = -2$.
   (f) Is the following integral equal to a finite number? Explain.
   $$\int_0^\infty \frac{e^{-2x}}{x^{1/4} + x} dx$$
   (g) Explain how the Jacobian can be used when computing a double integral.