Preliminary Exam – Calculus *January 2023*

Duration: 240 minutes
Instructions: Answer all questions, without the use of notes, books or calculators. A one-page crib sheet is allowed. Partial credit will be awarded for correct work. Each question is worth the same amount of points.

1. Short answer questions. Answer these in a few lines.

   (a) For vectors $\vec{a}$ and $\vec{b}$ that are neither parallel, nor perpendicular to each other, sketch the three vectors $\vec{a}$, $\vec{b}$, and $\text{Proj}_{\vec{a}}\,\vec{b}$ in two-dimensions.

   (b) Body Mass Index (BMI, $B(m,h)$) is a function of m (mass in kg) and h (height in m) whose ranges include underweight (BMI<18.5), normal weight (BMI: 18.5-24.9), overweight (BMI: 25-29.9) and obese (BMI>30). What do you think the signs of $\frac{\partial B}{\partial m}$ and $\frac{\partial B}{\partial h}$ are? Justify your answers.

   (c) A region R is bounded by the curves $y = \frac{4}{x}$, $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $y = \frac{3}{x^2}$. Find an appropriate change of variables u and v to evaluate the area of R. Find the Jacobian of that transformation.

   (d) Maria has evaluated the following integral:

   $$\int_{-2}^{2} \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_{-2}^{2} = -\frac{1}{2} - \left(\frac{1}{2}\right) = -1$$

   What did Maria do wrong?

   (e) Jimmy has made the following argument regarding divergence of a series: "Consider the infinite series $\sum\limits_{n=2}^{\infty} \frac{1}{n(n-1)}$. Since $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$, we can write the series as $\sum\limits_{n=2}^{\infty} \frac{1}{n-1} - \sum\limits_{n=2}^{\infty} \frac{1}{n}$. The first series, $\sum\limits_{n=2}^{\infty} \frac{1}{n-1}$ diveges due to the limit comparison test with $\frac{1}{n}$ and the second series, $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test. Since both series diverge, their difference also diverges. " What did Jimmy do wrong?

2. **True or False?** Justify your answer either by counterexample or explanation.

   (a) $\displaystyle\int_0^1 \int_x^1 f(x,y)dydx = \int_0^1 \int_y^1 f(x,y)dxdy$

   (b) It is possible for a continuous function $f(x,y)$ to have two local minima and no local maxima

   (c) If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ both do not converge, then $\sum\limits_{n=1}^{\infty} a_n b_n$ also does not converge

   (d) Given the parametric equations $x = f(t)$ and $y = g(t)$, if $\frac{dy}{dx} = \frac{dx}{dy}$, then $f(t) = g(t) + C$, where C is a constant.

   (e) If f and g are increasing on an interval I, then fg is increasing on I.

3. Evaluate the following limits or explain why the limit does not exist.

   (a) $\lim\limits_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$ (b) $\lim\limits_{n\to\infty} (e^n + 3n)^{2/n}$ (c) $\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{4x^2+y^2}$

4. Evaluate the following derivatives

   (a) $\frac{dy}{dx}$ if $4x^5 + \tan(y) = y^2 + 5x$ (b) $\frac{d}{dx}\left(\int_1^{\sqrt{x}} \sin(t)dt\right)$

5. Evaluate the following definite or indefinite integrals (do not worry about simplifying fractions in your evaluation)

   (a) $\int\limits_{-1}^{1} \frac{e^x}{e^x - 1} dx$ (b) $\int\limits_{0}^{\frac{\sqrt{3}}{2}} x^3\sqrt{1-x^2}dx$ (c) $\int \sin(\ln(t))dt$

6. Consider the function $f(x) = \frac{1}{2-x}$.
   (a) Find the Power Series Representation of $f(x)$ assuming it is expanded about $x = 0$
   (b) Find the Power Series Representation of $f(x)$ assuming it is expanded about $x = 1$
   (c) Are the radii/intervals of convergence for (a) and (b) the same? Why?

7. The function $f(x, y) = x^2 + 2xy + y^3$ represents the price of a sushi roll as a function of the price of rice per pound, x, and of the price of fish per ounce, y.
   (a) Find a tangent plane approximation to $f(x, y)$ at the point $(1, 2)$
   (b) Use your answer from a) to estimate the price of a sushi roll if the price of rice is $\$1.25$/pound and that of fish is $\$1.75$/ounce.
   (c) It turns out that the price of rice and the price of fish are time-dependent: the price of rice changes in time according to $x(t) = 1 + \sin t$ and that of fish according to $y(t) = t^3 - 2t^2 + 4$, **Use the chain rule** to calculate $\frac{df}{dt}$. Explain in words what $\frac{df}{dt}$ means.

8. Consider an object of density $f(x, y, z) = z(x^2 + y^2 + z^2)$ that is above the plane $z = 0$, below the cone $z = \sqrt{x^2 + y^2}$, and within the cylinder $x^2 + y^2 = 1$.
   (a) Sketch the object
   (b) Express the integral for the total mass in Cartesian coordinates. **Do not evaluate**.
   (c) Express the integral for the total mass in cylindrical coordinates. **Do not evaluate**.
   (d) Express the integral for the total mass in spherical coordinates. **Do not evaluate**.

9. Let $\vec{F} = \langle y^3, -x^3 \rangle$ be a vector field and C be the positively oriented circle of radius 2 centered at the origin.
   (a) Set up the integral $\int_C \vec{F} \cdot d\vec{r}$ using a parameterization of C.
   (b) Use Green's Theorem to set up the integral $\int_C \vec{F} \cdot d\vec{r}$.
   (c) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ using either (a) or (b)
   (d) Is $\vec{F}$ conservative? If so, find the potential function $f(x, y)$.

10. Consider the surface S (comprised of S_1 and S_2) of the solid V bounded by $z = x^2 + y^2$ and $z = 4$ (see picture on the right) and the vector field $\vec{F} = y^2 z^3 \mathbf{i} + 2yz\mathbf{j} + 4z^2\mathbf{k}$. Assume S is positively oriented.
    (a) Evaluate $\iint_{S_2} \vec{F} \cdot \hat{n} \, dS_2$.
    (b) Evaluate $\iint_S \vec{F} \cdot \hat{n} \, dS$ without parameterizing the surface.
    (c) Use your answers from (b) and (c) to deduce the value of $\iint_{S_1} \vec{F} \cdot \hat{n} \, dS_1$.