# Applied Mathematics Preliminary Exam: Calculus

**University of California, Merced**

**January 8, 2024, 9.00am-1.00pm**

**Instructions**

**Read the following instructions carefully:**

- Write your name on the front page of your exam.
- This is a closed book, closed notes exam. No phones or calculators are to be used during the exam.
- Write each problem on a separate page. Please make sure you clearly mark the problem you are working on (e.g., 1a).
- It is important to show your work for each problem. Credit will NOT be given for correct answers without justification. Also, partial credit will be given for incorrect answers if some of the work is correct.
- Clearly mark out (cross out) any work that you are not including in your answer and you do not want graded.
- Be sure to staple your exam at the end and hand it in.
- Good luck!

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1. Short answer questions. Answer these in a few lines.

   (a) True or False? If $f'(c) = 0$ there is a maximum or minimum at $x = c$. Justify your answer with an explanation (if True) or counterexample (if False).

   (b) If $f(x) = \int_1^{x^3} \frac{1}{1+\ln(t)} dt$ for $x \geq 1$, then what is $f'(2)$?

   (c) The tortoise versus the hare: The speed of the hare is given by the function $H(t)$ whereas the speed of the tortoise is $T(t)$ where $t$ is time measured in hours and the speed is measured in miles per hour. What does the area between $H(t)$ and $T(t)$ on the interval $0 \leq t \leq 1$ represent and what are its units?

   (d) There are two types of improper integrals. Provide an example of each type and explain why your example is improper.

   (e) True or False? $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}(t)] = 2\mathbf{u}'(t) \times \mathbf{u}(t)$. Justify your answer.

2. Evaluate the following limits or explain why the limit does not exist.

   (a) $\lim\limits_{x \to 0^+} \left( \frac{1}{x^2} - \frac{1}{\tan(x)} \right)$ (b) $\lim\limits_{n \to 0^+} n^n$ (c) $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$

3. Is it possible for a function to have a continuous first derivative but not have a continuous second derivative? If so, provide such an example. If not, explain why it is not possible.

4. Evaluate the following integrals

   (a) $\int_0^\infty te^{-t} dt$

   (b) $\int \frac{\cos(x)}{\sin^2(x) - \sin(x)} dx$

5. Find the value $k$ such that the following function is a probability density function (pdf). Recall that a function is a pdf if it is nonnegative over it's domain and $\int_{-\infty}^{\infty} f(t)dt = 1$.

$$f(t) = \begin{cases} 0 & t < 0 \\ \frac{kt}{\sqrt{16-t^2}} & 0 \leq t \leq 4 \\ 0 & t > 4 \end{cases}$$

6. Explain what is incorrect about each of the statements below. Then provide a correct explanation.

   (a) The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges by the ratio test.

   (b) The series $\sum\limits_{n=1}^{\infty} \frac{n}{1.05^n}$ diverges since the terms $0.95 + 1.81 + 2.59 + 3.29 + \ldots$ are increasing.

7. A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue per day from $x$ units of jogging shoes and $y$ units of cross-trainers is given by $R(x, y) = -5x^2 - 8y^2 - 2xy + 42x + 102y$ where $x$ and $y$ are in thousands of units. The factory is capable of producing at most 10,000 units per day. Find the values of $x$ and $y$ to maximize the total revenue.

8. Consider the volume of a bead that is outside the cylinder $x^2 + y^2 = 1$ but inside the sphere $x^2 + y^2 + z^2 = 4$.
   (a) Sketch the object
   (b) Express the integral for the total volume in Cartesian coordinates. **Do not evaluate**.
   (c) Express the integral for the total volume in cylindrical coordinates. **Do not evaluate**.
   (d) Express the integral for the total volume in spherical coordinates. **Do not evaluate**.

9. Find the work done by the force field $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ in moving a particle from the point (3,0,0) to the point $(0,\frac{\pi}{2},3)$ along
   (a) a straight line.
   (b) the helix $x = 3\cos t$, $y = t$, and $z = 3\sin t$.
   (c) Is the force field conservative? Why or why not?

10. The surface of a fish trap consists of a cone given by $x^2 + y^2 = z^2, 0 \leq z \leq 1$ and the circular top of the cone. The trap is placed in water which has a flow field given by $\mathbf{F} =< x - y, x + z, z - y >$. Assume the surface is positively oriented. Calculate the total volume of the fluid flowing through the fish trap per time unit:
    (a) Using the divergence theorem
    (b) Using direct parameterization(s) of the surface, $S$.