Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, January 2022

Instructions: This examination lasts 4 hours. Each problem is worth 20 points. While there are 8 problems, your total score will be calculated by adding up your 6 highest scores. Hence, the maximum total score is $6 \times 20 = 120$ points. Show explicitly the steps and calculations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded for relevant work.

1. (a) By solving $A\mathbf{x} = \mathbf{0}$, find a basis for the null space of A

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{1}$$

   (b) For $\mathbf{u} = \begin{bmatrix} 1 & 0 & 2 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$, let S be the linear subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}$ and $\mathbf{v}$ (i.e. $S = \text{Span}\{\mathbf{u}, \mathbf{v}\}$). Find a basis for the orthogonal complement $S^\perp$ of S.

2. In your TA discussion, where you review how to compute the eigenvalues and eigenvectors of a square matrix, you use the following two 2×2 matrices:

$$X = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}. \tag{2}$$

   Based on the results for X and Y, a student comes up with the following conjecture: *for any real $n \times n$ matrix, (1) all eigenvalues are real and (2) n linearly independent eigenvectors can be found.*

   (a) Find a counterexample to each part of the conjecture.

   (b) Can you give a correct statement for some class of matrices?

3. Let R be a rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta \in \mathbb{R}. \tag{3}$$

   (a) Express the following quantities in terms of θ:
   - the 1-norm $\|R\|_1$ of R,
   - the ∞-norm $\|R\|_\infty$ of R,
   - the 2-norm $\|R\|_2$ of R,
   - the spectral radius $\rho(R)$ of R.

   (b) Compare the magnitudes of $\|R\|_1$, $\|R\|_\infty$, $\|R\|_2$, and $\rho(R)$.

4. We want to solve the following linear recurrence relation using eigenvalues and eigenvectors:

$$a_{n+2} = 6a_{n+1} - 8a_n \quad \text{with } a_1 = 1 \text{ and } a_2 = 4. \tag{4}$$

   (a) Find A satisfying

$$\mathbf{x}_{n+1} = A\mathbf{x}_n, \quad \text{where } \mathbf{x} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}. \tag{5}$$

   (b) Diagonalize $A = PDP^{-1}$.

   (c) Using $\mathbf{x}_{n+1} = A^n\mathbf{x}_1$, express a_n in terms of n.

5.

$$P = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}. \tag{6}$$

(a) Express P as $P = \mathbf{v}\mathbf{v}^T$ for a vector $\mathbf{v}$.

(b) Can you give a physical interpretation of P as a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$: $\mathbf{x} \mapsto P\mathbf{x}$?

(c) What is P^{100}? (Hint: Use the result of either (a) or (b))

(d) What is the rank of P?

(e) Find a vector $\mathbf{u}$ such that $\mathbf{u}$ is orthogonal to the null space of P.

6. The singular value decomposition of A is given as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = U\Sigma V^T. \tag{7}$$

(a) What is the rank of A?

(b) Write down an orthornomal basis of the range space of A.

(c) Write down an orthonormal basis of the null space of A.

(d) Can we obtain other singular value decompositions of A using

$$\bar{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \tilde{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{8}$$

instead of U? If so, what kind of changes are needed for Σ or V in each case?

7. We want to find the *best* linear fit to the following points:

$$\{(-1, 0),\ (0, 0),\ (0, 1),\ (1, 2)\}. \tag{9}$$

(a) By constructing a normal system, find the least-squares fit $y = ax + b$.

(b) Plot the best linear fit with the data points.

(c) Briefly explain in which sense the least-squares fit is optimal.

8. Consider the three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \\ -1 \end{bmatrix}. \tag{10}$$

(a) Perform the Gram-Schmidt process to find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(b) Using the result of (a), find the following decomposition

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}. \tag{11}$$