Instructions: This examination lasts 4 hours. Each problem is worth 20 points. While there are 8 problems, your total score will be calculated by adding up your 6 highest scores. Hence, the maximum total score is $6 \times 20 = 120$ points. Show explicitly the steps and calculations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded for relevant work.

1. (a) By solving $Ax = 0$, find a basis for the null space of $A$

\[ A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \]  

(b) For $u = \begin{bmatrix} 1 & 0 & 2 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$, let $S$ be the linear subspace of $\mathbb{R}^4$ spanned by $u$ and $v$ (i.e. $S = \text{Span}\{u, v\}$). Find a basis for the orthogonal complement $S^\perp$ of $S$.

2. In your TA discussion, where you review how to compute the eigenvalues and eigenvectors of a square matrix, you use the following two $2 \times 2$ matrices:

\[ X = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}. \]  

Based on the results for $X$ and $Y$, a student comes up with the following conjecture: for any real $n \times n$ matrix, (1) all eigenvalues are real and (2) $n$ linearly independent eigenvectors can be found.

(a) Find a counterexample to each part of the conjecture.

(b) Can you give a correct statement for some class of matrices?

3. Let $R$ be a rotation matrix

\[ R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad \theta \in \mathbb{R}. \]  

(a) Express the following quantities in terms of $\theta$:
- the 1-norm $\|R\|_1$ of $R$,
- the $\infty$-norm $\|R\|_\infty$ of $R$,
- the 2-norm $\|R\|_2$ of $R$,
- the spectral radius $\rho(R)$ of $R$.

(b) Compare the magnitudes of $\|R\|_1$, $\|R\|_\infty$, $\|R\|_2$, and $\rho(R)$.

4. We want to solve the following linear recurrence relation using eigenvalues and eigenvectors:

\[ a_{n+2} = 6a_{n+1} - 8a_n \quad \text{with} \quad a_1 = 1 \text{ and } a_2 = 4. \]  

(a) Find $A$ satisfying

\[ x_{n+1} = A x_n, \quad \text{where} \quad x = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}. \]  

(b) Diagonalize $A = PDP^{-1}$.

(c) Using $x_{n+1} = A^n x_1$, express $a_n$ in terms of $n$. 
5. \( P = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \). 

(a) Express \( P \) as \( P = vv^T \) for a vector \( v \).

(b) Can you give a physical interpretation of \( P \) as a linear transformation \( \mathbb{R}^3 \to \mathbb{R}^3: \mathbf{x} \mapsto P\mathbf{x} \)?

(c) What is \( P^{100} \)? (Hint: Use the result of either (a) or (b))

(d) What is the rank of \( P \)?

(e) Find a vector \( u \) such that \( u \) is orthogonal to the null space of \( P \).

6. The singular value decomposition of \( A \) is given as follows:

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = U\Sigma V^T.
\]

(a) What is the rank of \( A \)?

(b) Write down an orthonormal basis of the range space of \( A \).

(c) Write down an orthonormal basis of the null space of \( A \).

(d) Can we obtain other singular value decompositions of \( A \) using \( \bar{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) or \( \bar{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \) instead of \( U \)? If so, what kind of changes are needed for \( \Sigma \) or \( V \) in each case?

7. We want to find the best linear fit to the following points:

\[ \{(−1, 0), (0, 0), (0, 1), (1, 2)\} \).

(a) By constructing a normal system, find the least-squares fit \( y = ax + b \).

(b) Plot the best linear fit with the data points.

(c) Briefly explain in which sense the least-squares fit is optimal.

8. Consider the three vectors

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}. \]

(a) Perform the Gram-Schmidt process to find an orthonormal basis \( \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} \) for the subspace spanned by \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).

(b) Using the result of (a), find the following decomposition

\[
[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3] \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{22} & r_{22} & r_{23} \\ r_{33} \end{bmatrix}.
\]