Applied Math Preliminary Exam: Linear Algebra

University of California, Merced, January 2023

Instructions: This examination lasts 4 hours. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work.

Problem 1. (a) Provide an example of a linear system with 3 equations and 2 unknowns with infinitely many solutions. Explain.

(b) Provide an example of a linear system with 3 equations and 3 unknowns with no solutions. Explain.

(c) Explain why a system of linear equations with 2 equations and 3 unknowns cannot have a unique solution.

Problem 2. For what values of q does the following matrix have rank (a) 1, (b) 2, (c) 3? Explain your reasoning in detail.

$$B = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$$

Problem 3. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}.$$

(a) Determine a basis for the column space of A.

(b) Determine a basis for the nullspace of A.

(c) Determine a solvability condition for $Ax = b$ where $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Problem 4. A linear tranformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has an eigenvector $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ with eigenvalue 1/4 and two eigenvectors, $\begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}$, both with eigenvalue 3. Find the image of the following vectors under the transformation T. Be sure to justify your findings.

$$\text{(i)} \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} \quad \text{and} \quad \text{(ii)} \begin{bmatrix} -1 \\ -2 \\ 12 \end{bmatrix}.$$

Problem 5. (a) Prove that if the non-zero vectors $v_1, v_2, \ldots, v_n$ are orthogonal to each other, then they are linearly independent.

(b) Prove that if $A \in \mathbb{R}^{n\times n}$ has orthonormal columns, then A has orthonormal rows as well.

Problem 6. Prove that λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ if and only if $\det(A - \lambda I) = 0$, where I is the $n \times n$ identity matrix.

Problem 7. Let $A = I + uv^\top$, where $u, v \in \mathbb{R}^n$ and I is the $n \times n$ identity matrix.

(a) What are the eigenvalues and the corresponding eigenvectors of A.

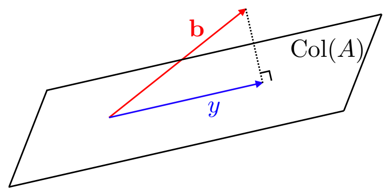
(b) Name a condition on u and v such that A is diagonalizable. Explain.

Problem 8. Let $A \in \mathbb{R}^{m\times n}$.

(a) Prove that $\mathrm{Col}(A)$ is a vector subspace of $\mathbb{R}^m$.

(b) Prove that if $b \in \mathrm{Col}(A)$ and $z \in \mathrm{Null}(A^\top)$, then $b \perp z$.

Problem 9. Let $A \in \mathbb{R}^{m\times n}$, where $m \geq n$ and A has full column rank. Explain why the orthogonal projection y of a vector b onto the column space of A



is given by $y = A(A^\top A)^{-1}A^\top b$.

Problem 10. Short answers.

(a) Determine whether the following statement is true or false. Explain your reasoning.

If no two vectors in a given set are scalar multiples of each other, then the set of vectors are linearly independent.

(b) Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Explain.