Applied Mathematics Preliminary Exam: Linear Algebra

University of California, Merced

January 12, 2024, 9.00am-1.00pm

Instructions

Read the following instructions carefully:

- Write your name on the front page of your exam.
- This is a closed book, closed notes exam. No phones or calculators are to be used during the exam.
- Write each problem on a separate page. Please make sure you clearly mark the problem you are working on (e.g., 1a).
- It is important to show your work for each problem. Credit will NOT be given for correct answers without justification. Also, partial credit will be given for incorrect answers if some of the work is correct.
- Clearly mark out (cross out) any work that you are not including in your answer and you do not want graded.
- Be sure to staple your exam at the end and hand it in.
- Good luck!

1. (5 points) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, and denote by $\mathbf{N}(\mathbf{A})$ and $\mathbf{C}(\mathbf{A})$ the nullspace and column space of $\mathbf{A}$, respectively.
   (a) Define the nullspace ($\mathbf{N}(\mathbf{A})$) and the columnspace ($\mathbf{C}(\mathbf{A})$) of this matrix.
   (b) Show that $\mathbf{N}(\mathbf{A})$ is a vector subspace of $\mathbb{R}^n$.
   (c) Show that $\mathbf{C}(\mathbf{A})$ is a vector subspace of $\mathbb{R}^m$.

2. (10 points) Let
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}.$$
   (a) Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$. *Hint: find the solution (or set of all solutions, if applicable) or show that there is no solution.*
   (b) Show that the pivot columns form a basis for $\mathbf{C}(\mathbf{A})$. What is the dimension of $\mathbf{C}(\mathbf{A})$?
   (c) Find a basis for $\mathbf{N}(\mathbf{A})$. (No need to show that the vectors you suggest form a basis.) What is the dimension of $\mathbf{N}(\mathbf{A})$?
   (d) Without deriving the nullspace and column space of $\mathbf{A}^\top$, can you tell what are the dimensions of $\mathbf{N}(\mathbf{A}^\top)$, $\mathbf{C}(\mathbf{A}^\top)$? Explain.

3. (10 points) Let
$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$
   (a) Does the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ have a solution? Explain.
   (b) What is the least squares solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$?
   (c) Find the projection $\mathbf{p}$ of the vector $\mathbf{b}$ onto the column space of $\mathbf{A}$.
   (d) Use Gram-Schmidt[1] to find the orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$, for the column space of $\mathbf{A}$.
   (e) Let us assume that the columns of $\mathbf{A}$ are $\mathbf{a}_1$ and $\mathbf{a}_2$. Write the columns of $\mathbf{A}$ using the $\{\mathbf{q}_1, \mathbf{q}_2\}$ basis. What are the coefficients? *Hint: Write each column as a linear combination of $\{\mathbf{q}_1, \mathbf{q}_2\}$ and find the coefficients.*
   (f) What is the $\mathbf{QR}$ decomposition of $\mathbf{A}$?
   (g) How and why would you incorporate the $\mathbf{QR}$ decomposition of $\mathbf{A}$ into a procedure to determine the least square solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$?

4. (5 points) If a diagonalizable matrix $\mathbf{A}$ (with real entries) has orthonormal eigenvectors and real eigenvalues must it be symmetric? Briefly explain why or give a counterexample.

5. (10 points) Let
$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}.$$

[1] Recall, the Gram-Schmidt process starts with independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and (typically) ends with orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$. At step j it subtracts from $\mathbf{a}_j$ its components in the directions $\mathbf{q}_1, \ldots, \mathbf{q}_{j-1}$ that are already settled, i.e., $\mathbf{A}_j = \mathbf{a}_j - (\mathbf{q}_1^\top \mathbf{a}_j)\mathbf{q}_1 - \cdots - (\mathbf{q}_{j-1}^\top \mathbf{a}_j)\mathbf{q}_{j-1}$. Then $\mathbf{q}_j$ is the unit vector $\mathbf{A}_j / \|\mathbf{A}_j\|$, when $\mathbf{A}_j \neq 0$.

(a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

(b) Write the vector $\mathbf{u}(0) = [2 \quad 0]^\top$ as a combination of the eigenvectors.

(c) Solve the equation $\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$ with initial condition $\mathbf{u}(0)$ given in (b). In other words, write the solution $\mathbf{u}(t)$ in terms of the eigenvectors of $\mathbf{A}$.

(d) Find the Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$.

6. (10 points) Prove the following properties:

(a) Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors. Prove that if this set of vectors includes the zero vector then this set of vectors is not linearly independent.

(b) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be non-zero vectors that are orthogonal to each other. Prove that they are linearly independent.

(c) Let λ_1 and λ_2 be two distinct eigenvalues of an $n \times n$ matrix $\mathbf{A}$, i.e., $\lambda_1 \neq \lambda_2$. Prove that the corresponding eigenvectors are linearly independent.

(d) For an $n \times m$ matrix $\mathbf{A}$, prove that every $\mathbf{y} \in \mathbf{N}(\mathbf{A}^\top)$ is orthogonal to every vector $\mathbf{b}$ in $\mathbf{C}(\mathbf{A})$.