Applied Math Preliminary Exam: Linear Algebra
University of California, Merced, January 2025

Instructions: This examination lasts 4 hours. Each problem is worth 15 points. While there are 10 problems, your total score will be calculated by adding up your 8 highest scores. Hence, the maximum total score is $8 \times 15 = 120$ points. Show explicitly steps and computations in your solutions. Credit will not be given to answers without explanation. Partial credit will be awarded to relevant work.

1. (a) Provide an example of a linear system with 3 equations and 3 unknowns with infinitely many solutions. Explain.

   (b) Provide an example of a linear system with 2 equations and 3 unknowns with no solutions. Explain.

   (c) Suppose x^* is a solution to the linear system $Ax = b$. Explain why this solution is unique if A has linearly independent columns.

2. Determine all values of b and q such that the following linear system has at least one solution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & b & 0 \\ 2 & 4 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} q \\ 1 \\ 4 \end{bmatrix}. \tag{1}$$

3. (a) Let

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2}$$

   Compute the eigenvalues of Q.

   (b) Prove that the eigenvalues of a real orthogonal matrix have modulus one.[1]

   (c) Find all the elements of the set $\mathcal{S}$ given by

$$\mathcal{S} = \left\{ C \in \mathbb{R}^{3\times 3} \colon C \text{ is orthogonal, symmetric, and positive definite} \right\}. \tag{3}$$

4. Let $A = I + ww^\top$, where w is a column vector in $\mathbb{R}^n$ with $\|w\|_2 = 1$ and I is the $n \times n$ identity matrix.

   (a) Prove that A is invertible, with

$$A^{-1} = I - \frac{1}{2} ww^\top. \tag{4}$$

   (b) What are the eigenvalues and the corresponding eigenvectors of A^{-1}? Explain.

5. A Toeplitz matrix A is a matrix whose elements satisfy the following property: $A_{i,j} = A_{i+1,j+1}$. An example of a 4×4 Toeplitz matrix is the following:

$$\begin{bmatrix} 1 & -3 & 0 & 4 \\ \pi & 1 & -3 & 0 \\ 2 & \pi & 1 & -3 \\ 8 & 2 & \pi & 1 \end{bmatrix}. \tag{5}$$

   Prove that the set of 4×4 Toeplitz matrices forms a vector space.

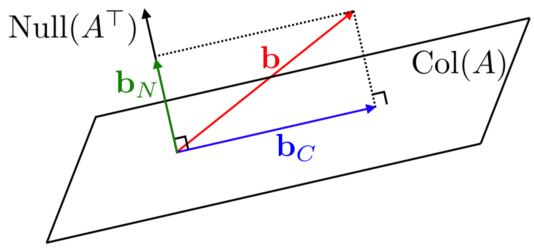
6. Prove that if v_1, v_2 are non-zero column vectors in $\mathbb{R}^2$ and are linearly independent, then that the 2×2 matrix

$$M = v_1 v_1^\top + v_2 v_2^\top \tag{6}$$

   is invertible.

[1] Note that the modulus of a complex number $a + bi$ ($a, b \in \mathbb{R}$) is defined as $\sqrt{a^2 + b^2}$.

7. Let $A \in \mathbb{R}^{m\times n}$, where $m \geq n$ and A has full column rank.



(a) Derive the expression for the orthogonal projection $\mathbf{b}_C$ of a vector $\mathbf{b}$ onto the column space of A, which is given by

$$\mathbf{b}_C = A(A^\top A)^{-1}A^\top \mathbf{b}. \tag{7}$$

(b) Derive the expression for the orthogonal projection $\mathbf{b}_N$ of a vector $\mathbf{b}$ onto the null space of $A^\top$, which is given by

$$\mathbf{b}_N = (I - A(A^\top A)^{-1}A^\top)\mathbf{b}. \tag{8}$$

8. Let $x, y \in \mathbb{R}^n$ be non-zero vectors.

(a) By considering the inequality

$$0 \leq \left\| \frac{x}{\|x\|_2} - \frac{y}{\|y\|_2} \right\|_2^2, \tag{9}$$

prove the Cauchy-Schwarz inequality for the 2-norm: $x^\top y \leq \|x\|_2\|y\|_2$.

(b) Using part (a), prove that the vector 2-norm satisfies the triangle inequality, i.e.,

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2. \tag{10}$$

(c) Using part (b), prove that the matrix 2-norm also satisfies the triangle inequality, i.e., if A and B are $n \times n$ matrices, then

$$\|A + B\|_2 \leq \|A\|_2 + \|B\|_2, \tag{11}$$

where the matrix 2-norm is defined as

$$\|C\|_2 = \max_{z\neq 0} \frac{\|Cz\|_2}{\|z\|_2}. \tag{12}$$

9. Let $A \in \mathbb{R}^{m\times n}$.

(a) Prove that if $p \in \text{Range}(A)$ and $q \in \text{Null}(A^\top)$, then $p^\top q = 0$.

(b) Prove that if $b \in \text{Range}(A)$ and $b \in \text{Null}(A^\top)$, then $b = 0$.

(c) Prove that if $z \in \text{Null}(A^\top A)$, then $z \in \text{Null}(A)$. (**Hint:** Let $b = Az$ and use part (b).)

10. Short answers. Determine whether the following statements are true or false. Explain your reasoning.

(a) If all of the eigenvalues of a matrix Z are 0, then Z is the zero matrix.

(b) If A and B are invertible $n \times n$ matrices, then their product AB is also invertible.

(c) Let w_1, w_2 be two linearly independent vectors in $\mathbb{R}^3$. Then $\{w_1, w_2\}$ is a basis for $\mathbb{R}^2$.