Differential Equations Preliminary Exam *January 2023*

Duration: 240 minutes
Instructions: Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. (3 pts) What is your name? (Failure to answer this question will be a cause for concern.)
2. (12 pts) Describe all the possible behaviors as $t \to \infty$ of solutions $x(t)$ of the equation below as k varies over the real numbers.
$$\frac{dx}{dt} = x(k - x^2)$$
3. (8 pts) Find a solution to the initial value problem given below (an implicit solution is acceptable)
$$\frac{dx}{dt} = \frac{(x - x^2)t}{3 - x} \quad \text{with} \quad x(0) = \frac{1}{2}.$$
4. (11 pts: 3,3,5) Consider the equation of a forced spring-mass system, where $x(t)$ is the mass's position as a function of time
$$\frac{d^2x}{dt^2} + \omega^2 x = \sin \alpha t$$
   (a) Find the general solution of this system when $\omega \neq \alpha$, call it $x_\alpha(t)$.
   (b) Find the general solution of this system when $\omega = \alpha$, call it $x_\omega(t)$.
   (c) Assume that $x(0) = 0$ and $x'(0) = 0$ for both $\alpha = \omega$ and $\alpha \neq \omega$. Verify that
$$\lim_{\alpha \to \omega} x_\alpha(t) = x_\omega(t).$$
5. (27 pts: 3,3,3,3,6,6,3 pts) Let $x(t)$ represent the population of kangaroos and $y(t)$ the population of rabbits, both as functions of time. Suppose that their population dynamics are described the equations below, where $\alpha \geq 0$ is a parameter.
$$\begin{aligned} \frac{dx}{dt} &= x(4 - 2x - \alpha y) \\ \frac{dy}{dt} &= y(9 - 3\alpha x - 3y) \end{aligned}$$
   (a) In the absence of rabbits ($y = 0$), show how the population of kangaroos evolves over time by plotting approximate solutions in the xt-plane.
   (b) Describe in words the interactions between the species when $\alpha = 0$.
   (c) Describe in words the meaning of α in terms of interactions between the species.
   (d) Find all the equilibrium points of the given system when $\alpha \neq 0$.
$$\text{Hint:} \quad \begin{aligned} 2x + \alpha y &= 4 \\ 3\alpha x + 3y &= 9 \end{aligned}$$
$$\text{has solution} \quad x = \frac{3\alpha - 4}{\alpha^2 - 2} \text{ and } y = \frac{4\alpha - 6}{\alpha^2 - 2}$$
   (e) For $\alpha = 1$, give a phase portrait for the evolution of both populations and describe the long-term behavior of the system if the initial populations of rabbits and kangaroos are not zero.
   (f) For $\alpha = 2$, give a phase portrait for the evolution of both populations and describe the long-term behavior of the system if the initial populations of rabbits and kangaroos are not zero.

(g) Explain why the results you found above make sense based on your interpretation of the meaning of α.

6. (15 pts) Find two independent solutions of the equation below in the form of infinite Series

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{9}\right) y = 0.$$

7. (16 pts: 3,3,5,5) The temperature of a ton of inactive nuclear waste as a function of time is given by $N(t)$, where t is computed in hours. This waste is placed outside where the ambient temperature is a known function $A(t)$. The rate of change of the temperature of the nuclear waste is proportional to the difference between the outside air and itself, with a proportionality constant $k > 0$.

   (a) Give an equation describing the rate of change of the temperature of the nuclear waste and provide the units of all the quantities involved.

   (b) Using the units of k, give an interpretation of its magnitude.

   (c) If the outside temperature in degree Celsius is $A(t) = 25 + 10 \sin\left(\frac{\pi t}{12}\right)$, solve for the temperature of the nuclear waste over time if its initial temperature is N_0.

   (d) Describe in words the short and long time behaviors of this system.

8. (8 pts: 4,4) Consider the system of equations below.

$$\begin{aligned} \frac{dx}{dt} &= x(1-y) \\ \frac{dy}{dt} &= y(2x-4) \end{aligned}$$

   (a) Show that $C = y + 2x - \ln(yx^4)$ is a conserved quantity of the system.

   (b) Argue that for a fixed $C > 0$, the portion of a curve $C = y+2x-\ln(yx^4)$ lying in the first quadrant is bounded (i.e. does not go to infinity).